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THE ORIGIN AND NATURE OF COLOR IN PLANTS.

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A list of the more important papers published, up until within the past ten years, on the subject of plant colors is given in Dippel's *Das Mikroskop.*[1] Of these the papers by Pringsheim[2] on the examination of chlorophyl and related substances, and by Müller[3] on the spectrum-analysis of the color substances of flowers, are probably the most important.

Pringsheim confined his attention mainly to a spectroscopic study of chlorophyl and the yellow substances in germinating plants, yellow flowers and yellow autumn leaves. He concluded that the yellow substances from these several sources were but modifications of chlorophyl. The yellow principle found in germinating plants he regarded as closely related to chlorophyl, and the yellow substance in autumn leaves as a more remote modification of it. He did not consider, however, as subsequent writers have claimed, that these substances were identical.

Two years before the appearance of Pringsheim's paper, Kraus[4] stated that he had separated from an alcoholic solution of chlorophyl by means of benzol two distinct substances, one yellow and the other blue, the latter being taken up by the benzol. Pringsheim, however, showed that the blue substance was in reality chlorophyl, and that the alcoholic solution, which showed faint chlorophyl-like bands in the spectroscope, still contained some chlorophyl.

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While Pringsheim believed that there were two modifications of chlorophyl, one yellow and the other green, the former predominating in germinating plants grown in the dark, and the latter or green substance in leaves exposed to the light, still he did not believe that they could be separated from each other by the method proposed by Kraus.

Yet notwithstanding Pringsheim's well-founded criticisms of the method employed by Kraus, and taking for granted that there were two principles composing chlorophyl, nearly all investigators since Kraus's work was published have practically employed his method as modified by Hansen[8] for the separation of the so-called yellow and green chlorophyl. According to this method of Hansen, fresh material is extracted with 95 per cent. alcohol, the liquid filtered, and to the filtrate 30 to 50 per cent. of water is added; the solution is shaken with petroleum ether and the liquids separated, the ether taking up the green substance, or chlorophyl proper, and the hydro-alcoholic solution holding the yellow principle.

If autumn leaves are treated in the same way, the ether solution will contain very little chlorophyl, while the hydro-alcoholic solution will contain a yellowish or reddish substance, depending upon the kind of material examined. It has usually been considered that this yellow substance in autumn leaves is associated in summer with the active plastids, and on account of its having little food value remains behind. It has furthermore been considered by many that the yellow principle in young leaves is identical with that in autumn leaves and the yellow substance found in yellow flowers, fruits and roots.

Kinds of Colors in Plants.

Colors in plants may be considered to be due to definite constituents which either themselves are colored or produce colors when acted upon by other substances. These substances are found in all parts of the plant, and apparently in all of the cells excepting certain meristematic or dividing cells. They may be divided into two well-differentiated classes, namely, (1) those which are associated with the plastids, or organized bodies in the cell, and (2) those which occur in the cell-sap, or liquid of the cell.

So-called White Colors.

The so-called white colors in plants do not properly belong to either class, but may be said to be appearances rather, due to the absence of color, and depending upon the reflection of light from transparent cells separated by relatively large intercellular spaces containing air. In other words the effect produced by these cells may be likened to that produced by the globules in an emulsion. The white appearance is most pronounced in the pith cells of roots and stems, where on the death of the cells the size of the intercellular spaces is increased and the colorless bodies in the cells as well as the walls reflect the light like snow crystals.

Methods of Extraction.

During this investigation I have examined by means of the Leitz micro-spectroscope the various kinds of coloring substances to which I shall refer but, except in the case of chlorophyl, did not obtain results which were entirely satisfactory, and will endeavor to give special attention to this phase of the subject in another paper. It is frequently difficult to extract and isolate these substances in a sufficiently pure condition for spectroscopic work, particularly as many of them change rapidly.

In this paper, therefore, I shall confine myself to the consideration of the behavior of the extracted coloring substances toward chemical reagents.

The material containing the coloring matter was in all cases separated as nearly as possible from that which was free from color or contained it in less amount. Various solvents were used in the extraction of the coloring substances, depending upon the solubility or nature of the substance. The solvent mostly employed was alcohol (95 per cent.), in some cases dilute alcohol (50 per cent.) or water (hot or cold) was employed.

The plastid colors were extracted by placing the fresh material in 95 per cent. alcohol and allowing it to macerate in the dark for a day or two. I usually took the precaution to tear the material with the fingers rather than to cut it. The solution so obtained contains other than the plastid coloring substances, which latter may be isolated in a more or less pure condition by either of the following methods: (1) The alcohol is distilled off and the solution evaporated on a water bath to near dryness, boiling water is then added and

the solution filtered, the extract washed with hot water until the filtrate is colorless; the extract is then taken up with cold alcohol. (2) In the other method the alcoholic solution is diluted with water; and ether, benzin, benzol, xylol, or other similar solvent is added, and the mixture shaken in a separatory funnel. The ethereal layer containing the plastid color may be further purified by shaking it in a separatory funnel with alcohol, adding sufficient water to cause separation of the two layers. The ethereal solution is then distilled and evaporated on a water bath to near dryness, and the pigment taken up with cold alcohol. In either case the alcoholic solution may be boiled for an hour or two with zinc in a reflux condenser, whereby the more or less oxidized plastid pigments are restored. This is a particularly important procedure in the microspectroscopic examination of chlorophyl, and may be used as a means of detecting chlorophyl in other substances.

In order to obtain the coloring principles in early leaves, as the red coloring principle in the leaves of oak, rose, etc., it was found most satisfactory to extract the material with alcohol, add xylol or similar solvent, and then sufficient water to effect separation of the solutions, using a separatory funnel. The cell-sap color remains in the hydro-alcoholic solution, and the traces of xylol should be removed by heating the solution on a water-bath, as the presence of xylol causes a cloudiness in the solution on the addition of the reagents to be subsequently employed.

The cell-sap colors of flowers, as of pansy, tulip, etc., are separated from the plastid pigments in the same way as just mentioned in connection with early leaves.

The cell-sap colors in fall leaves are easily removed by treating the more or less comminuted material with hot or cold water.

In some cases there are several associated colors, and these may be extracted separately by taking advantage of their varying solubility, as in the case of carthamus, where the red principle is extracted with water and the yellow principle with alcohol.

In still other cases special methods are employed, as in the extraction of carotin from carrot according to the method proposed by Husemann.[6] The grated carrot is mixed with water, squeezed through cheese-cloth, and a small quantity of dilute sulphuric acid and tannin added to the mixture, forming a coagulum which settles to the bottom of the precipitating jar. The supernatant liquid is removed by means of a syphon and the coagulum treated six or

seven times with 80 per cent. alcohol, which removes mannit and hydro-carotin; the coagulum is then extracted with hot carbon disulphide, which removes the carotin. This solution is evaporated to about half the original volume, an equal amount of absolute alcohol added, and set aside to crystallize, the carotin separating.

One of the striking observations made during this investigation was that in the case of the cell-sap colors the solution was different in color, as compared to the natural color, or sometimes almost colorless, reagents, however, striking colors as intense or even more intense than the original colors.

For the convenience of those who may wish to follow similar studies, the plants which I examined may be grouped according to the solvents which I found best adapted for the extraction of the coloring substances. There is also given the part of the plant employed and the color of the solutions I obtained.

Color Principles Extracted with Alcohol.

Name of Plant.	*Part Used.*	*Color of Solution.*
1. Apple (Baldwin) (*Pyrus Malus*)	Epicarp	Light yellowish-red
2. Apple (Bellefleur) (*Pyrus Malus*)	Epicarp	Pale yellow
3. Arbutus (*Epigæa repens*)	Petals	Pale straw
4. Azalea (*Azalea nudiflora*)	Petals	Pale straw
5. Beet (*Beta vulgaris*)	Leaves	Deep green
6. Blackberry (*Rubus Canadensis*)	Stems	Reddish-brown
7. Buttercup (*Ranunculus acris*)	Petals	Deep yellow
8. Cabbage, red (*Brassica oleracea*)	Leaves	Purplish-red
9. Capsicum (*Capsicum fastigiatum*)	Dried fruit	Yellowish-red
10. Carnation, red (*Dianthus Caryophyllus*)	Petals	Deep red
11. Carrot (*Daucus Carota*)	Root	Deep reddish-yellow
12. Celery (*Apium graveolens*)	Etiolated leaves	Bright greenish-yellow
13. Chondrus (*Chondrus crispus*)	Fronds	Light yellowish-green
14. Cinquefoil (*Potentilla Canadensis*)	Petals	Greenish-yellow
15. Cranberry (*Oxycoccus macrocarpus*)	Fruit	Deep red
16. Daffodil (*Narcissus Pseudo-Narcissus*)	Petals	Deep yellow
17. Dandelion (*Taraxacum officinale*)	Petals	Lemon-yellow
18. Dock (*Rumex crispus*)	Spring leaves	Reddish-brown
19. Dogwood (*Cornus Florida*)	Fruit	Brownish-yellow
20. Dulce (*Rhodymenia palmata*)	Fronds	Light yellowish-green
21. Elder (*Sambucus Canadensis*)	Spring leaves	Reddish-brown
22. Fucus (*Fucus vesiculosus*)	Fronds	Greenish-brown
23. Hepatica (*Hepatica triloba*)	Petals	Lemon-yellow or greenish-yellow
23a. Hepatica (*Hepatica triloba*)	Involucre	Purplish-red
24. Iris (*Iris versicolor*)	Petals	Violet
25. Jack-in-the-pulpit (*Arisæma triphyllum*)	Spathe	Purplish-red
26. Japanese quince (*Cydonia Japonica*)	Petals	Bright purplish-red
26a. Lemon peel	Epicarp	Yellow
27. Mallow (*Malva sylvestris*)	Petals	Violet
28. Maple (*Acer rubrum*)	Flowers	Yellowish or brownish-red
29. Marigold (*Calendula officinalis*)	Petals	Deep yellow
30. Oak, red (*Quercus coccinea*?)	Spring leaves	Reddish-brown
30a. Orange peel	Epicarp	Orange-yellow
31. Pansy, blue (*Viola tricolor*)	Petals	Purplish-red

Color Principles Extracted with Alcohol—*Continued.*

Name of Plant.	*Part Used.*	*Color of Solution.*
32. Pansy, yellow (*Viola tricolor*)	Petals	Deep yellow
33. Pineappie (*Ananas sativa*)	Outer portion	Brown
34. Radish (*Raphanus Raphanistrum*)	Purplish layer of root	Light red
35. Rose (*Rosa gallica*)	Dried petals	Light brown
35a. Rose (*Rosa* ——)	Early leaves	Reddish-brown
36. Safflower (*Carthamus tinctorius*)	Petals	Deep yellow
37. Saffron (*Crocus sativus*)	Dried stigmas	Yellowish-red
38. Skunk cabbage (*Spathyema fatida*)	Green leaves	Deep green
39. Skunk cabbage (*Spathyema fatida*)	Inner portion of leaf	Deep yellow
40. Skunk cabbage (*Spathyema fatida*)	Spathe [buds	Deep yellowish-red
41. Skunk cabbage (*Spathyema fatida*)	Scales	Purplish-red
42. Skunk cabbage (*Spathyema fatida*)	Tips of leaf buds	Yellowish-red
43. Spinach (*Spinacea oleracea*)	Leaves	Deep green
44. Sweet Cicely (*Washingtonia Claytoni*)	Spring leaves	Reddish-brown
45. Tomato (*Lycopersicon esculentum*)	Fruit	Pale yellow
46. Tulip (*Tulipa Gesneriana*)	Petals	Light reddish-brown
47. Turnip (*Brassica napus*)	Purplish layer of root	Pale yellow
48. Violet, blue (*Viola cuculata*)	Petals	Pale purplish-red
49. Violet, yellow (*Viola scabriuscula*)	Petals	Yellow
50. Wahoo (*Euonymus Americanus*)	Winter leaves	Reddish-brown

Color Principles Extracted with Dilute Alcohol.

51. Black Mexican corn (*Zea Mays*)	Grains	Light purplish-red
52. Geranium, house (*Pelargonium* ——)	Petals	Light purplish-red
53. Geranium, wild (*Geranium maculatum*)	Petals	Pale straw
54. Houstonia (*Houstonia cærulea*)	Petals	Pale straw
55. Hyacinth, dark red (*Muscari botryoides*)	Petals	Light yellowish-red
56. Hyacinth, blue (*Muscari botryoides*)	Petals	Purplish-red
57. Lilac (*Syringa vulgaris*)	Petals	Brownish-yellow
58. Rhubarb (*Rheum* ——)	Outer portion of petioles	Pale red
59. Strawberry (*Fragaria* ——)	Fruit	Yellowish-red
60. Violet, blue (*Viola cuculata*)	Petals	Greenish-yellow
61. Wistaria (*Kraunhia frutescens*)	Petals	Pale brown

Color Principles Extracted with Water.

62. Beech (*Fagus Americana*)	Autumn leaves	Reddish-yellow
63. Beet (*Beta vulgaris*)	Root	Deep red
64. Blackberry (*Rubus Canadensis*)	Outer portion of stems	Brownish-red
65. Blackberry (*Rubus Canadensis*)	Fruit	Purplish-red
66. Cranberry (*Oxycoccus macrocarpus*)	Fruit	Deep red
67. Dogwood (*Cornus Florida*)	Autumn leaves	Reddish-brown
67a. Dulce (*Rhodymenia palmata*)	Fronds	Purplish
68. Elder (*Sambucus Canadensis*)	Dried fruit	Purplish-red
69. Grape (*Vitis vinifera*)	Fruit	Purplish-red
70. Holly (*Ilex Aquifolium*)	Fruit	Deep brownish-red
71. Hydrangea (*Hydrangea Hortensia*)	Neutral flowers	Brownish-red
72. Indian cucumber (*Medeola Virginiana*)	Autumn leaves	Deep brownish-red
73. Mallow (*Malva sylvestris*)	Petals	Dark purplish-red
74. Maple (*Acer saccharum*)	Autumn leaves	Brownish-red
75. Marigold (*Calendula officinalis*)	Dried petals	Deep brownish-red
76. Oak, white (*Quercus alba*)	Autumn leaves	Brownish-red
77. Rhubarb (*Rheum* ——)	Outer portion of petioles	Pale red
78. Rose, wild (*Rosa* ——)	Pericarp	Deep brownish-red
79. Safflower (*Carthamus tinctorius*)	Dried petals	Deep brownish-red
80. Saffron (*Crocus sativus*)	Dried stigmas	Deep yellowish-red
81. Solomon's seal (*Vagnera racemosa*)	Fruit	Deep red

Plastid Color Substances.

The green color in plants is due, as is well known by botanists, to a green pigment known as chlorophyl which is associated with a plastid or organized protoplasmic body, forming a so-called chloroplast. Chlorophyl is distinguished from all other plant substances by possessing a dark broad band between the Fraunhofer lines A and C at the red end of the spectrum, which is apparent even in very dilute solutions. It also shows in more concentrated solutions a broad band extending from F to the violet end of the spectrum, a narrow band between C and D, or the orange portion of the spectrum, and two narrow bands between D and E, or the yellow portion of the spectrum.

Pringsheim examined spectroscopically solutions of the yellow substances found in etiolated germinating leaves, and also the yellow substances of yellow flowers and autumn leaves, and observed the characteristic chlorophyl bands only by using tubes more than three hundred millimeters thick. Inasmuch as small tubes holding five or ten cubic centimeters are sufficient for the examination of chlorophyl, by means of the Zeiss or Leitz microspectroscope, and also because a dilute solution is necessary, one is surprised that Pringsheim and others have used tubes of such enormous thickness, and that they concluded from the more or less indistinct bands which they observed that these substances were modifications of chlorophyl. It is not at all unlikely that what he actually had were concentrated solutions of as many different principles, each of which contained traces of chlorophyl, notwithstanding the care he exercised in separating the green and yellow portions in the material which he used.

In my own studies on the yellow principle of developing leaves I used the buds of skunk cabbage, which develop under ground and under leaves and are of considerable size before exposed to light. The outer light greenish-yellow portions were removed, and only the intense yellow central portion used. This material was extracted in the dark with alcohol. The solution thus obtained is of a pure lemon-yellow color, and may be freed from cell-sap substances either by evaporation to an extract, washing with water, dissolving in cold alcohol, and then boiling with zinc; or by treating the original alcoholic solution with petroleum benzin, whereby the pure yellow leaf substance is separated from the cell-sap substance.

This yellow principle is combined with plastids, which are about one micron in diameter, being spherical or polygonal in shape, and lying closely packed in the palisade cells of both the upper and lower surfaces of the leaf. The yellow plastids are distinguished from the leucoplastids, which occur in the epidermal and mesophyl cells, as well as the chloroplastids, which are found later in the green leaves, by being smaller, relatively more numerous and by not manufacturing either reserve or assimilation starch. The associated pigment is further distinguished from chlorophyl by not being fluorescent; in having a broad band extending from 65 to the red end of the spectrum, and another extending from 50–52 to the violet end of the spectrum, when examined by means of the Leitz micro-spectroscope; and in being less soluble in alcohol and more so in benzin than chlorophyl. This latter characteristic affords a means of partially separating it from chlorophyl, and for this principle I propose the name *etiophyl*, and for the associated plastid, which seems to be a distinct body, I propose a corresponding name, *etioplast*, these terms being used expressly for the purpose of avoiding confusion. The etioplasts completely pack the cells in which they are found, and may be regarded as meristematic plastids, which later give rise to the chloroplastids.

The yellow color in certain roots, flowers and fruits is apparently in all cases due to a yellow pigment associated with a plastid known as a chromoplast. These plastids are distinguished from the other plastids by being of variable shape and in usually containing protein grains. The associated pigment resembles in some respects etiophyl and chlorophyl, in that it is more or less soluble in ether, benzol, xylol, carbon disulphide, etc. These pigments, for the most part, appear to be unaffected by either mineral or organic acids, but usually give some shade of green with alkalies, potassium cyanide, sodium phosphate or iron salts. In some cases they are affected by alum, iodine, sodium nitrite, or sodium nitrite and sulphuric acid, as given in Table I.[1]

[1] In the examination of plant colors the following reagents were found useful: Sulphuric acid, 10 per cent.; hydrochloric acid, 10 per cent.; nitric acid, 10 per cent.; citric acid, 5 per cent.; oxalic acid, 5 per cent.; sodium hydrate, 10 per cent.; ammonium hydrate, 10 per cent.; potassium cyanide, 1 per cent.; sodium phosphate, 5 per cent.; ferric chloride, 3 per cent.; ferrous sulphate, 2.5 per cent.; hydrogen peroxide, 3 per cent.; salicylic acid, saturated solution, gallic acid, 1 per cent.; sodium nitrite, 1 per cent.; sodium nitrite followed by sulphuric

Inasmuch as there seems to be a class of these principles which are distinguished by their solubility, as well as reactions with various chemicals, I venture to propose the name *chromophyl* for these yellowish or orange-colored pigments.

All of the coloring substances given in Table I are soluble in xylol, ether and similar solvents, as well as alcohol, but are sparingly soluble in water.

There are several substances which behave much like the plastid substances, but which are insoluble in xylol, ether, etc., and appear to occupy an intermediate position between the true plastid color substances and the cell-sap colors. I have therefore placed them in class by themselves in Table II.

Cell-sap Color Substances.

During the course of metabolism the plant cell manufactures other color substances which are not combined with the protoplasm, but which are contained in the cell-sap, or liquid of the cell. These substances, unlike the plastid colors, are insoluble in xylol, ether and similar solvents, but are soluble in water and alcohol, which affords a means of separating them from the plastid colors. These cell-sap pigments may occur in cells free from plastids or in the vacuoles of cells containing plastids, but not associated with them as a part of the organized body or plastid. They are usually extracted along with the chlorophyl and remain in the hydro-alcoholic solution after separation of the plastid pigment by means of xylol or other solvent. These pigments have one property in common with the chromophyl substances, namely, with alkalies, potassium cyanide and sodium phosphate, they assume some shade of green. They are distinguished, however, by the fact that the colors are markedly affected by acids and alkalies and by iron salts. They are in most cases also affected by other reagents, as shown in the accompanying tables. These substances being so sensitive to reagents, probably accounts for the various shades and tints characteristic not only of flowers but of leaves as well. My observations on the germinating kernels of black Mexican corn show that even in contiguous cells the constituents associated with the dye

acid; potash alum, 10 per cent.; ammonio-ferric alum, 5 per cent.; iodine solution containing .1 per cent. iodine and 0.5 per cent. potassium iodide; tannin, 3 per cent.

vary to such an extent that the pigment in one cell is colored reddish, in another bluish-green, and in another purplish.

The results of the examination of the cell-sap colors are given in Tables III, IV and V, and while it might seem a very easy matter to divide plant colors into reds, blues and purples, it will be seen that this is almost impracticable, and that the colors given in these tables merge into one another.

An examination of the color substances found in early spring leaves and in autumn leaves showed that these substances are in the nature of cell-sap colors, behaving toward reagents much like the cell-sap colors of flowers, and indeed in some instances they are apparently identical, as will be seen by comparing the results given in Table VI with those given in Tables III, IV and V.

Conclusions.

1. The white appearance in flowers and other parts of plants is due to the reflection and refraction of light in more or less colorless cells separated usually by large intercellular spaces containing air.

2. The green color of plants is due to a distinct pigment, chlorophyl, contained in a chloroplastid, and appears to be more or less constant in composition in all plants. The chloroplastid is furthermore characterized by usually containing starch.

3. The yellow color substance in roots, flowers and fruits is due to a pigment, to which I have given the name chromophyl. This substance is contained in a chromoplastid which varies considerably in shape, and usually contains proteid substances in addition.

4. In the inner protected leaf-buds there is a yellow principle which I have termed etiophyl, and which is contained in an organized body which I have termed an etioplast. The etioplast does not appear to contain either starch or proteid substances.

5. The blue, purple and red color substances in flowers are dissolved in the cell-sap, and are distinguished for the most part from the plastid colors by being insoluble in ether, xylol, benzol, chloroform, carbon disulphide and similar solvents, but soluble in water or alcohol. While quite sensitive to reagents yet none of these colors behave precisely alike.

6. Cell-sap color substances corresponding to the cell-sap colors of flowers are also found in early or spring leaves and in autumn leaves.

In addition I desire to say that I am inclined to look upon the chromoplastids of both flowers and fruits as having the special function of manufacturing or storing nitrogenous food materials, for the use of the developing embryo or developing seed, particularly as protein grains are usually contained in them. The same may be said of the chromoplasts in roots, as in carrot, where the proteids of the chromoplasts are utilized by the plant of the second year.

I am further inclined to consider the cell-sap colors, like other unorganized cell-contents, as alkaloids, volatile oils, etc., to be incident to physiological activity, and of secondary importance in the attraction of insects for the fertilization of the flower and dispersal of the seed.

Finally, I acknowledge my indebtedness to Miss Florence Yaple, Philadelphia, for valuable assistance in the preparation of this paper.

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I. Examination of Plastid Color Substances.

	Etiophyl		Chlorophyl		Chromophyl					
	39. *Skunk cabbage*	12. *Celery*	43. *Spinach*	38. *Skunk cabbage*	16. *Daffodil*	7. *Buttercup*	14. *Cinquefoil*	32. *Yellow pansy*	49. *Yellow violet*	29. *Marigold*
Mineral acids .	Cloudy*	Cloudy	Cloudy	Pale brown, cloudy	Cloudy, color less intense	Cloudy	Paler and slightly cloudy	Decolorized, cloudy	Slightly cloudy	No effect
Organic acids .	Cloudy	No effect	Cloudy	Light brownish-green	Cloudy, color less intense	Slightly cloudy	Paler and slightly cloudy	Decolorized, cloudy	Slightly cloudy	No effect
Alkalies	O.c.† intensified	Yellowish-green	No effect	No effect	Slightly green	No effect	Yellowish-green	Yellowish-green	Yel'sh-green, becoming colorless	Light yellowish-green
Potassium cyanide	O. c. intensified	Yellowish-green	No effect	No effect	Slightly green	No effect	Yellowish-green	Yellowish-green	Yellowish-green	Light yellowish-green
Sodium phosphate	O. c. intensified	Yellowish-green	No effect	No effect	Slightly green	No effect	Yellowish-green	Yellowish-green	Yellowish-green	Light yellowish-green
Ferric chloride.	Light olive-green	Brown	No effect	Brownish-green	Light olive-gr'n, slightly cloudy	Yellowish-brown, cloudy	Brownish-green	Olive-green	Yellowish-green	Greenish-brown
Ferrous sulphate	Light olive-green, cloudy	Pale brown	Light brown	Brownish-green	Light olive-gr'n, slightly cloudy	Yellowish-green, cloudy	Pale green	Green, becoming olive-green	Yellowish-green, cloudy	Pale greenish-brown
Salicylic acid. .	Remains clear	No effect	No effect	No effect	No effect	No effect	Slightly decolorized	Decolorized	Partly decolorized	No effect
Gallic acid. . .	Remains clear	No effect	No effect	No effect	No effect	No effect	Slightly decolorized	Decolorized	Partly decolorized	No effect
Hydrogen peroxide	No effect	No effect	No effect	No effect	No effect	No effect	Slightly decolorized	Decolorized	Partly decolorized	No effect
Sodium nitrite .	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect	Faint brown	No effect
Sodium nitrite and sulphuric acid	Cloudy	Cloudy	Light bluish-green	Pale brown, cloudy	Decolorized	Decolorized	Decolorized	Faint brown	Decolorized	No effect
Alum	Cloudy	Slightly cloudy	Cloudy	Pale brownish-green, cloudy	Cloudy	Cloudy	No effect	Pale green, cloudy	Pale green, cloudy	No effect
Ammonio-ferric alum	Olive-green	Brown	Greenish-brown	Brownish-green, cloudy	Olive-green	Yellowish-brown	Gr'n-brown, changing to brown	Brownish-green	Pale yellowish-brown	Greenish-brown
Iodine solution.	No effect	No effect	No effect	Greenish-brown	Slightly cloudy	Pure green, distinct	No effect	Brownish	No effect	No effect
Tannin	No effect	No effect	No effect	No effect	Slightly cloudy	No effect	No effect	Slightly decolorized	No effect	No effect

*The cloudy appearance of solutions recorded in this table is probably due to an oily or resinous substance associated with the coloring principle, or to the fact that the coloring principle is insoluble in much diluted alcohol.

† O. c., original color of solution.

I. Examination of Plastid Color Substances—*Continued.*

	Chromophyl								
	75. *Marigold*	17. *Dandelion*	46. *Tulip*	26a. *Lemon peel*	30a. *Orange peel*	45. *Tomato*	11. *Carrot*	25. *Jack-in-the-pulpit*	78. *Wild rose hips*
Mineral acids . . .	No effect	Cloudy	Slightly cloudy	No effect	No effect	Cloudy	Cloudy	No effect	Slightly cloudy
Organic acids . . .	No effect	Slightly cloudy	Slightly cloudy	No effect	No effect	Cloudy	Slightly cloudy	No effect	Slightly cloudy
Alkalies	O. c. slightly intensified	No effect	Faint yellowish-green	Pure yellow	Greenish-yellow	Very faint green	Slightly greenish	No effect	No effect
Potassium cyanide	O. c. slightly intensified	No effect	Faint yellowish-green	Pure yellow	Greenish-yellow	Very faint green	Slightly greenish	No effect	No effect
Sodium phosphate	O. c. slightly intensified	No effect	Faint yellowish-green	Pure yellow	Light greenish-yellow	Very faint green	Slightly greenish	No effect	No effect
Ferric chloride . .	Olive-green, changing to brown	Greenish-yellow	Greenish-yellow	Dark brown	Yellowish-brown	Yellowish-green	Very light brown	No effect	Light greenish-brown
Ferrous sulphate .	Pale olive-green	Cloudy, pale yellowish-green	Faint green	Dark brown	Greenish-yellow	Pale green	No effect	No effect	Light greenish-brown
Salicylic acid . . .	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect
Gallic acid	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect
Hydrogen peroxide	Pale yellow	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect
Sodium nitrite. . .	Pale yellow	Slightly cloudy	No effect	No effect	No effect	Decolorized	No effect	No effect	No effect
Sodium nitrite and sulphuric acid. .	No effect	Decolorized	Decolorized	Light brown	No effect	Decolorized	Decolorized, bluish fluorescence	Decolorized	Decolorized
Alum	No effect	Slightly cloudy	No effect	Pale green	No effect	Decolorized	Slightly cloudy	No effect	No effect
Ammonio-ferric alum	Dark yellowish-brown	Greenish-yellow, cloudy	Brownish-yellow	Deep greenish-brown	Light yellowish-brown	Yellowish-brown	No effect	No effect	Light greenish-brown
Iodine solution . .	No effect	No effect	No effect	No effect	No effect	Faint green	Pure green, or yellow with green fluorescence	No effect	No effect
Tannin	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect	No effect

II. Examination of Intermediate Color Substances.

	37, 80. *Saffron*	36. *Safflower*	79. *Safflower*	2. *Apple (Bellefleur)**	70. *Holly*
Mineral acids	No effect	Cloudy	No effect	No effect	Light yellow
Organic acids	No effect	Cloudy	No effect	No effect	Slightly decolorized
Alkalies	No effect	Greenish-yellow	Darkened slightly	Greenish-yellow	Brown
Potassium cyanide	No effect	Greenish-yellow	Darkened slightly	Greenish-yellow	No effect
Sodium phosphate	No effect	Greenish-yellow	Darkened slightly	Pale yellow	No effect
Ferric chloride	Darkened or greenish-brown	Light olive-green to light brown	Dark greenish-brown	Green, changing to olive-green	Light greenish-brown
Ferrous sulphate	No effect	Light olive-green to light brown	Light greenish-brown	Pale green	Light greenish-brown
Salicylic acid	No effect	No effect	No effect	No effect	No effect
Gallic acid	No effect	No effect	No effect	No effect	No effect
Hydrogen peroxide	No effect	No effect	No effect	No effect	No effect
Sodium nitrite	No effect	No effect	No effect	Pale brown	No effect
Sodium nitrite followed by sulphuric acid	Pale yellow	Cloudy	No effect	Light brown	No effect
Alum	No effect	No effect	No effect	No effect	No effect
Ammonio-ferric alum	Darkened or yellowish-brown	Light yellowish-brown	Deep olive-brown	Greenish to greenish-brown	Greenish-brown
Iodine solution	No effect	No effect	No effect	No effect	No effect
Tannin	No effect	No effect	No effect	No effect	No effect

* Cloudy on addition of water to alcoholic solution.

III. Examination of Blue Cell-sap Color Substances.

III. Examination of Blue Cell-sap Color Substances.

	23. *Hepatica*	48. *Violet, blue*	31. *Pansy, blue*	56. *Hyacinth, blue*	61. *Wistaria*	54. *Houstonia*	27. *Mallow flowers*	*Litmus solution*	24. *Iris*
Natural color . . .	Blue	Violet-blue	Purple	Purplish-blue	Light blue	Light blue	Dark blue	Deep purple	Purple to violet
Mineral acids . . .	Pale yellowish-red	Pure deep red	Intense rich red	Intense rich red	Purplish-red	Light yellowish-red	Deep purplish-red	Yellowish-red	Pure deep red
Organic acids . . .	Pale yellowish-red	Pure red	Purplish-red	Violet-red	Pale purplish-red	Light yellowish-red	Deep purplish-red	Yellowish-red	Pure deep red
Alkalies	Green	Green	Green to brownish-green	Light brownish-green	Yellowish-green	Yellowish-green	Brownish-green	Pure blue	Green, changing to yellowish-green
Potassium cyanide	Green	Green	Green	Green	Yellowish-green	Yellowish-green	Brownish-green	Pure blue	Green
Sodium phosphate	Pale green	Green	Green	Green	Yellowish-green	Yellowish-green	Green	Purplish-blue	Green
Ferric chloride . .	Olive-green	Olive-green	Intense blue	Purplish-brown to brown	Olive-green	Deep olive-green	Brownish-green	Purplish-red	Purplish-blue, changing to brown
Ferrous sulphate .	Light olive-green	Bluish-green	Deep blue	Blue	Brownish-purple	Olive-green	Reddish-brown	Purplish-red	Pure blue
Salicylic acid . . .	Faint yellowish-red	Faint red	O. c. intensified	No effect	Pale reddish	Slightly reddened	No change	Yellowish-red	Faint red
Gallic acid	Faint yellowish-red	Slight red	O. c. intensified	O. c. intensified	No effect	No effect	No effect	Yellowish-red	Faint red
Hydrogen peroxide	No effect	Slight red	O. c. intensified	No effect	No effect	No effect	No effect	Yellowish-red	No effect
Sodium nitrite . .	No effect	Green	Pure green	No effect	Slightly greenish	Light green	Pale purplish	No effect	Decolorized
Sodium nitrite, followed with sulphuric acid . .	Pale yellowish-brown	Red, becoming decolorized	Red, then colorless	Yellowish-red	Pale reddish	Light yellowish-red	Golden yellow	Yellowish-red	Faint red, almost decolorized
Alum	Slightly yellowish-gr'n	Gobelin-blue	Sky-blue, light blue	Decolorized	No effect	No effect	No effect	Yellowish-red	Pure blue, distinct
Ammonio-ferric alum	Olive-green	Greenish-brown	Deep blue, rapidly changing to bluish-gr'n	Reddish-brown	Olive-green	Olive-green	Greenish-brown	Yellowish-red	Purplish, changing to brown
Iodine and potassium iodide . . .	No effect	No effect	Pale yellowish-red	No effect	No effect	No effect	No effect	Blue	Decolorized
Tannin	No effect	No effect	Reddened	No effect	No effect	No effect	No effect	Purplish-red	No effect

IV. Examination of Purple Cell-sap Color Substances.

	25. *Jack-in-the-pulpit*	40. *Skunk cabbage*	41. *Skunk cabbage*	23a. *Hepatica involucres*	57. *Lilac*	51. *Black Mexican corn*	68. *Elderberries*
Natural color	Violet-red	Purplish-red	Purplish-red	Purplish	Purple	Purplish	Purplish
Mineral acids . . .	Pure deep red	Deep red	Red	Faint salmon	Somewhat cloudy	Pure red	Purplish-red
Organic acids . . .	Pure deep red	Light purplish-red	Light red	Faint salmon	Somewhat cloudy	Pure red	Purplish-red
Alkalies	Green	Intense green	Green	Yellowish-green	Greenish, changing to yellowish-brown	Bluish-green	Pure green
Potassium cyanide.	Green	Intense green	Green	Yellowish-green	Greenish, changing to yellowish-gr'n	Bluish-green	Pure green
Sodium phosphate .	Green	Green	Green	Pale yellowish-green	Greenish, changing to yellowish-gr'n	Light bluish-green	Light green
Ferric chloride . . .	Purplish-red, changing to brown	Dark purple	Purplish-red	Pale greenish-brown	Deep brownish-green	Greenish-brown	Pale greenish-brown
Ferrous sulphate . .	Violet	Dark purple	Purplish-red	Very pale greenish-brown	Faint olive-green	Purple	Pale purplish
Salicylic acid	Pure red	No effect	No effect	No effect	No effect	Pinkish	O. c. slightly intensified
Gallic acid	Pure red	No effect	No effect	No effect	No effect	Slightly pink	O. c. slightly intensified
Hydrogen peroxide.	Pure red	No effect	No effect	No effect	No effect	Red	No effect
Sodium nitrite . . .	No effect	No effect	Becoming cloudy	No effect	No effect	No effect	Pale brown
Sodium nitrite, followed by sulphuric acid	Pure deep red	Yellowish-red	Yellowish-red or orange	Pale yellowish-brown	Pale yellow	Yellowish-red	Purplish-red
Alum	Violet	Purplish-red, fluorescent	Faint purplish-red	No effect	No effect	Red, changing to violet	No effect
Ammonio-ferric alum	Purplish-brown	Greenish-brown	Purplish-green	Greenish-brown	Deep brownish-green	Yellowish-brown	Olive-green
Iodine solution . . .	No effect	No effect	No effect	No effect	No effect	No effect	No effect
Tannin	Slightly red	No effect	No effect	No effect	No effect	Faint pink	No effect

IV. EXAMINATION OF PURPLE CELL-SAP COLOR SUBSTANCES—*Continued.*

	61. *Blackberry*	69. *Concord grapes*	8. *Red cabbage*	47. *Turnip*	71. *Hydrangea*	67a. *Dulce*	53. *Wild geranium*
Natural color . . .	Reddish-purple	Bluish-purple	Purplish-red	Purplish-red	Reddish-purple	Purplish-red	Light purplish-red
Mineral acids . . .	Purplish-red	O. c. intensified	Rose-red	Purplish-red	Yellowish-red	Purple, losing fluorescence	Deep red
Organic acids . .	Purplish-red	O. c. intensified	Light rose-red	Light purplish-red	Yellowish-red	Purple, losing fluorescence	Faint red
Alkalies	Brownish-purple	Pure green, changing to olive-green	Intense green	Light green	Yellowish-green	Pale yellowish-brown	Yellowish-green, changing to yellowish-brown
Potassium cyanide.	Brownish-purple	Pale bluish-green	Intense green	Light green	Yellowish-green	Yellowish-brown	Greenish-yellow
Sodium phosphate .	Slightly changed	Violet	Bluish-green	Light green	Yellowish-green	No effect	Greenish-yellow
Ferric chloride . . .	Purplish-brown, changing to brown	Greenish-brown	Rose-purple	Light greenish-brown	Olive-green	Purplish-brown	Deep olive-green
Ferrous sulphate . .	Purple	Purplish-brown	Purple	Light blue	Olive-green	No effect	Blue
Salicylic acid	No effect	No effect	O. c. intensified	No effect	Slightly redden'd	No effect	Red
Gallic acid	No effect	No effect	O. c. intensified	Slightly pink	Slightly redden'd	Slightly purple	No effect
Hydrogen peroxide	No effect	No effect	No effect	No effect	No effect	No effect	No effect
Sodium nitrite . . .	Purple color intensified	Purplish-brown	No effect	No effect	Light greenish	No effect	No effect
Sodium nitrite, followed by sulphuric acid	Brownish-red	Brown	Yellowish-red	Pale yellowish to brownish-red	Brownish-yellow	Faint purple, losing fluorescence	Deep red
Alum	No effect	No effect	Violet, purplish-red	Faint violet	No effect	No effect	No effect
Ammonio-ferric alum	Deep purple	Olive-green	Purplish-brown	Pale brown	Olive-green	Purplish-brownish red	Bluish-brown green
Iodine solution .	No effect	No effect	No effect	No effect	No effect	No effect	Red, becoming colorless
Tannin	No effect	No effect	No effect	No effect	No effect	O.c.renderedbluish	No effect

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V. Examination of Red Cell-sap Color Substances.

	52. *House geranium*	46. *Tulip*	10. *Carnation*	55. *Hyacinth*	35. *Rose*	4. *Azalea*	28. *Maple flowers*	15. *Cranberry*
Natural color . . .	Very deep red	Reddish	Deep red	Red	Red	Pink	Red	Deep red
Mineral acids . . .	Deep orange-red	Purplish-red	Yellowish-red	Deep orange-red	Rose red	Pale red	Yellowish-red, cloudy	No change
Organic acids . . .	Pale orange-red	Purplish-red	O. c. intensified	O. c. intensified	Light purplish-red	Pale red	Yellowish-red, clear	No change
Alkalies	Yellowish-green, changing to brown	Green	Olive-green, changing to red'sh-brown	Green, changing to brown	Green, changing to brown	Light greenish-yellow	Dark green	Brownish-purple
Potassium cyanide.	Yellowish-green	Green	Olive green	Green	Green	Light greenish-yellow	Green, changing to brown	Brownish-purple
Sodium phosphate.	No effect	Light green	Light green	Green	Green	Pale greenish-yellow	Yellowish-green	Brownish-purple
Ferric chloride . .	Purplish, changing to greenish brown	Purplish-br'wn to brown	Brownish-red	Deep red	Greenish-black	Greenish-yellow	Olive-green	Brownish-red
Ferrous sulphate .	Deep violet	Deep brownish-red	Brownish-red	Pinkish changing to wine-red	Bluish-black	Pale purplish-brown	Deep blue	Reddish-brown
Salicylic acid . . .	Yellowish-red	Purplish-red	O. c. intensified	O. c. intensified	O. c. intensified	Faint pink	Yellowish-red	No effect
Gallic acid.	Yellowish-red	Purplish-red	O. c. intensified	O. c. intensified	O. c. intensified	Faint pink	Yellowish-red	No effect
Hydrogen peroxide	Yellowish-red	Light purplish-red	O. c. intensified	O. c. intensified	O. c. intensified	No effect	Faint yellowish-red	No effect
Sodium nitrite . .	No effect	Light brownish-red	Slightly decolorized	No effect	Slightly decolorized	No effect	Light greenish-yellow	Yellowish-brown
Sodium nitrite, followed with sulphuric acid . . .	Deep orange-red	Light brown	O. c. restored and intensified	Deep orange-red	Yellowish-red	Pale red	Yellowish-red, cloudy	Yellowish-brown
Alum	Pale red	Purplish-red	Yellowish-red	Yellowish-red	Pale yellow, changing to reddish-brown with a greenish fluorescence	Faint pink	Purplish-red, cloudy	No effect
Ammonio-ferric alum	Deep violet, changing to purplish-brown	Brown	Brownish-red	Deep red	Dark olive-green	Pale greenish-brown	Deep blue	Brown
Iodine solution . .	Red, becoming decolorized	No effect	No effect	No effect	Yellowish-red	No effect	No effect	No effect
Tannin	Faint red	Purplish-red	No effect	No effect	No effect	No effect	No effect	No effect

V. EXAMINATION OF RED CELL-SAP COLOR SUBSTANCES—*Continued.*

	59. *Strawberry*	81. *Solomon's seal*	58, 77. *Rhubarb*	34. *Radish*	63. *Beet*	1. *Baldwin apple*	33. *Pineapple*
Natural color.	Yellowish-red to red	Ruby red	Pinkish to red	Bright red	Deep red	Red	Slightly reddened
Mineral acids	O. c. intensified	O. c. intensified	O. c. much intensified	Yellowish-red	No effect	O. c. intensified	Apparently no effect
Organic acids.	O. c. slightly intensified	O. c. intensified	O. c. intensified	O. c. much intensified	No effect	O. c. somewhat intensified	Pale yellowish-green
Alkalies	Brownish	Greenish-brown	Light green	Green	Green	Greenish-yellow	Pale yellowish-green
Potassium cyanide . . .	Purplish	Greenish-brown	Light green	Slight blue	Green	Greenish-yellow	Apparently no effect
Sodium phosphate . . .	Pale purple	Greenish-brown	Almost decolorized	Slight blue, then decolorized	Greenish	Yellowish-green	Yellowish-green
Ferric chloride	Reddish-brown	Reddish-brown	Brownish	Red	Greenish-brown	Deep green	Pale yellowish-green
Ferrous sulphate	Brownish-red	Light reddish-brown	Reddish	Red, slightly cloudy	Light greenish-brown	Brownish-green	No effect
Salicylic acid.	No effect	No effect	No effect	O. c. intensified	No effect	No effect	No effect
Gallic acid	No effect	No effect	No effect	O. c. strongly intensified	No effect	No effect	No effect
Hydrogen peroxide . .	No effect	No effect	No effect	O. c. intensified	No effect	No effect	No effect
Sodium nitrite	Slightly brown	No effect	Decolorized	No effect	No effect	No effect	No effect
Sodium nitrite, followed with sulphuric acid .	Pale yellowish-brown	Light reddish-brown	Pale yellow	Bright red	No effect	Brownish-red or brick-red	Light brown
Alum.	O. c. intensified	No effect	Purplish-red	Pale yellowish-red	No effect	No effect	No effect
Ammonio-ferric alum .	Brown	Brownish-red	Pale brown	Deep yellow or yellowish-brown	Dark brown	Olive-green	Pale greenish-brown
Iodine solution	No effect	No effect	No effect	No effect	No effect	No effect	No effect
Tannin.	No effect	No effect	No effect	No effect	No effect	No effect	No effect

VI. Examination of Leaf Coloring Principles.

	35. *Rose*	42. *Skunk cabbage*	30. *Oak*	6. *Blackberry*	21. *Elder*	18. *Dock*	50. *Wahoo*
Natural color	Greenish-red	Purplish-red	Deep red	Brownish-red	Greenish-red	Greenish-red	Greenish-red
Mineral acids	Rose color	Yellowish-red	Orange-red	Pale brownish-yellow	Light purplish-red	Slightly reddish	Yellowish-red
Organic acids	Rose color	Light yellowish-red	Orange-red	Pale brownish-yellow	Pale yellowish-red	No effect	Slightly reddened
Alkalies	Green, changing to greenish-yellow	Green	Greenish-brown	O. c. intensified	Intense yellowish-green	Yellowish-green	Greenish
Potassium cyanide	Pale green	Green	Greenish-brown	O. c. intensified	Intense green	Yellowish-green	Brownish-green
Sodium phosphate	Pale greenish-yellow	Green	Pale greenish-brown	O. c. slightly intensified	Intense green	Yellowish-green	Yellowish-green
Ferric chloride	Olive-green	Dark brownish-green	Bluish, changing to purplish	Olive-green, changing to brown	Deep green	Dark green	Olive-green
Ferrous sulphate	Deep blue	Dark brownish-green	Indigo blue	Pale olive-green	Deep green	Dark green	Faint orange-green
Salicylic acid	Pale red	No effect	Slightly orange-red	No effect	Faint yellowish-red	No effect	No effect
Gallic acid	Pale red	No effect	Slightly orange-red	No effect	Faint yellowish-red	No effect	No effect
Hydrogen peroxide	No effect	No effect	Slightly orange-red	No effect	Faint yellowish-red.	No effect	No effect
Sodium nitrite	No effect	No effect	No effect	No effect	Slightly green	No effect	Slightly green
Sodium nitrite, followed with sulphuric acid	Yellowish-brown	Brownish-yellow	Reddish-brown	No effect	Reddish, changing to brown	*Deep brownish-red*	Yellowish-red
Alum	Reddish-brown	No effect	Purplish or violet colored	No effect	Pale yellowish-brown	Pale yellowish-green	No effect
Ammonio-ferric alum	Olive-green	Purplish-green	Bluish-brown	Brownish-green	Deep green	Dark green	Olive-green
Iodine solution	No effect	No effect	Green, changing to purplish-brown	No effect	No effect	No effect	No effect
Tannin	Faint purplish-red	No effect	No effect	No effect	No effect	No effect	No effect

VI. Examination of Leaf Coloring Principles—*Continued.*

	44. *Sweet cicely*	74. *Maple*	76. *Oak*	67. *Dogwood*	72. *Indian cucumber*	62. *Beech.*
Natural color	Greenish-red	Dark red	Dark red	Dark red	Reddish	Greenish to brownish-yellow
Mineral acids	Faint yellowish-red	Yellowish-red	Yellowish-red	Yellowish-red	Deep purplish-red	Partly decolorized
Organic acids	Faint yellowish-red	Slightly yellowish-red	Slightly yellowish-red	Yellowish-red	Deep purplish-red	Partly decolorized
Alkalies	Yellowish	Olive-green, with ammonia, reddish brown	Reddish-brown	Brown	Green	Brown
Potassium cyanide	Yellowish	Deep brownish-red	Purplish-red	Brown	Green	Very light brown
Sodium phosphate	Yellowish	Light olive-green	Greenish-brown	Light brown	Light green	Faint brown
Ferric chloride	Olive-green	Deep blue precipitate	Blue, changing rapidly to olive-green	Deep blue, changing to olive-green	Reddish-brown	Greenish-brown
Ferrous sulphate	Olive-green	Deep blue solution	Blue	Deep blue	Reddish-brown	Pale greenish-brown
Salicylic acid	No effect	Slightly yellowish-red	No effect	Yellowish-red	Purplish-red	Partly decolorized
Gallic acid	No effect	Slight effect	No effect	Yellowish-red	Purplish-red	Partly decolorized
Hydrogen peroxide	No effect	No effect	No effect	No effect	Purplish-red	Partly decolorized
Sodium nitrite	Brownish	No effect	No effect	Brownish-yellow	Faint brown	No effect
Sodium nitrite, followed with sulphuric acid	Yellowish-brown	Yellowish-red	Yellowish-red	Yellowish-red, changing to yellowish-brown	Faint yellowish-brown	Light brown
Alum	Light greenish-yellow	No effect	No effect	Purplish-red	Purplish-red	Partly decolorized
Ammonio-ferric alum	Olive-green	Deep blue	Blue, changing to olive-green	Deep blue, changing to olive-green	Purplish-brown	Greenish-brown
Iodine solution	No effect	No effect	No effect	No effect	No effect	No effect
Tannin	No effect	No effect	No effect	No effect	Purplish-red	No effect

THE HISTORICAL USE OF THE RELATIVE PRONOUNS IN ENGLISH LITERATURE.

BY PROFESSOR WATERMAN T. HEWETT, PH.D.

(Read April 9, 1904.)

In examining the manuscript of a new volume submitted for publication, I was struck with the fact that the relative pronoun *which* was not used by the author. The question arose, whether there was a portion of our country in which, through historical or possibly educational influence, the use of *that* prevailed in place of *which*. In my subsequent reading, I marked the use of these pronouns in order to determine their literary use. Many of the characteristics of literary form depend upon the choice of the pronoun adopted. The use of one or the other pronoun is a characteristic of the style of representative English writers and lends a special quality to their form and expression.

The Germanic languages did not possess a distinctive relative pronoun. The place of such pronoun in Old English was supplied by *sē*, *sēo* and *þæt*, also by the indeclinable demonstrative form *þē* (the), which was frequently added to the article, and, though less frequently, by the interrogatives *which* and *who*. *What* (*hwæt*) as a relative occurs first at the beginning of the thirteenth century. Following the Conquest, the use of *þē* (the) as a relative declined, due, possibly, to the increasing tendency to use this particle in place of all the forms of the definite article. About 1200, the neuter *þæt* was, in general, used as a relative in both numbers and in all persons and genders in the nominative and accusative cases. This use may have been promoted by the influence of the French conjunction *que*.

The interrogatives *who* and *which* were used, but only in isolated cases, as relatives, *who* referring mainly to persons and *which* to things. By the time of the translation of the King James version of the Bible, in 1611, the development in the use of the relative pronouns had attained certain distinct features. The most striking differentiation in use consisted in the fact that *that* was made to refer to pronouns and *which* to nouns. The use of *which* had constantly increased and had gradually displaced *that*, and *who* and *what* had gained in frequency of use. The

present tendency in literature is to employ *who* and *which* at the expense of the earlier *that*.

Every scholar will judge from his own use, or from the environment in which his speech has been formed, in respect to the frequency and naturalness of the use of *which* and *that* in his own case. That which we do instinctively is the test of familiar expression. Writers upon the use of language in rhetorics and popular grammars exhibit great diversity of judgment respecting the use of these pronouns. Dean Alford, in his book upon the Queen's English, fourth edition, 1874, in speaking of the use of *who* and *which*, remarks: "Now we do not commonly use either one or the other of these pronouns, but make the more convenient one *that* make duty for both. We do not say 'The man *who* met me, nor the cattle *which* I saw grazing,' but 'The man *that* met me, the cattle *that* I saw grazing.'"

Bain, in his *Higher English Grammar*, says that *who* and *which* are most commonly preferred for co-ordination, but that they may also be used as restrictives. "However, *that* is the proper restrictive, explicative or defining relative. It would be a clear gain to confine *who* and *which* to co-ordination and to reserve *that* for the restrictive use alone. In the sentence 'His conduct surprised his English friends *who* had not known him long,' we mean either that his English friends generally were surprised (the relative being in this case co-ordinating), or that only a portion of them—namely, the particular portion that had not known him long—were surprised. The doubt would be removed by writing thus, 'His English friends *that* had not known him long.' So, also, in the sentence 'The next winter *which* you will spend in town will give you opportunity to make a more prudent choice;' this may either mean you will spend next winter in town or the next of the winters when you are to live in town, let that come when it may. In the former case *which* is the proper relative, and in the latter case *that*." According to my own impression, the ambiguity in the sentence "His English friends *that* had not known him long" would not be removed, as the author thinks, by the substitution of *that* for *which* in this case.

Genung, in *The Working Principles of Rhetoric*, 1902, says: "Typically, the relatives *who* and *which* assume that the antecedent is fully defined in sense, their office being to introduce additional information about it. They may accordingly be called

the additive relative, and are equivalent to a demonstrative with a conjunction, 'and he,' 'and this,' 'and these.' The relative *that* assumes that its antecedent is not yet fully defined, its office being to complete or restrict its meaning. It may accordingly be called the restrictive relative, and may generally be represented, by way of equivalent, by an adjectival or participial phrase."

Professor Hill, of Harvard, says: "Few good authors observe the rule that *who* or *which* should be confined to cases in which the relative clause explains the meaning of the antecedent or adds something to it, and *that* to cases in which the relative clause restricts the meaning of the antecedent. This rule, however helpful to clearness it may be in theory, few good authors observe; considerations of euphony prevent adoption of the rule" (*Principles of Rhetoric*, revised and enlarged, page 136).

Meiklejohn, in his *English Language*, says: "*That* is generally employed to limit, distinguish and define. Thus we say 'The house *that* I built is for sale.' Here, the word *that* is an adjective limiting or defining the noun house. Hence, it may be called the defining relative. *Who* or *which* introduces a new fact about the antecedent; *that* only marks it off from the other nouns."

We thus have here representative opinions from English, Scotch and American scholars, who base their judgment mainly upon their practical experience of language and not upon an examination of the literary monuments in different periods. It is our purpose, therefore, to ascertain the historical use of these pronouns and to determine the frequency with which they occur in representative works in literature, since the period of Wiclif's translation of the Bible.

An examination of the two texts of Layamon's *Brut*, issued about seventy years apart, show how complete the distinction between these pronouns had become in that period. In the older text (of about 1205) the earlier relatives of different genders as well as *þe* are used, while the later manuscript B. (of about 1275) represents these pronouns by a uniform *þaet* (that).

A.

(Line 13,827) An alle mine liue, þe ich, iluued habbe.

B.

In al mine liue, þat ich ileued habbe.

In the century which follows, *who* and *which* occur, but less frequently. In the language of Chaucer (1340–1400), *that* is the

prevailing relative; *who*, *whose* and *whom* occur but in few instances, and may then relate either to persons or things, as in Shakespeare. Chaucer stood more under French influence as regards language than his great contemporary, Wiclif (1324–1384), who in his translation of the Bible was influenced more by Latin constructions. If we examine the *Morte d'Arthur* of Sir Thomas Malory (1400–1470), which lies intermediate in time between Wiclif and Tyndale, we find in 555 lines 30 cases of the use of *that* as a relative, 6 cases of the use of *who* or *whom* in indirect questions, or as an indefinite relative equal to *whoever*, while *which* (*the whiche*) occurs but once in the nominative and once, "*for the whiche*," governed by a preposition. This shows that *that* retained its supremacy in the fifteenth century.

If we now compare the use of the relative pronouns in Wiclif's (1384) and in Tyndale's (1526) translations of the Gospels, which are separated by about a century and a half, we find the following results.

The approximate number of times that the relative pronouns *which*, *that* and *who* occur in the four Gospels in the Wiclif and Tyndale versions is as follows:

In Wiclif's version of the Gospels

which occurs 29 times in Matthew
18 " " Mark
97 " " Luke
27 " " John
171 " " the four Gospels

In Tyndale's version

which occurs 135 times in Matthew
61 " " Mark
241 " " Luke
125 " " John
562 " " the four Gospels

In Wiclif's version

that occurs 205 times in Matthew
84 " " Mark
284 " " Luke
228 " " John
801 " " the four Gospels

In Tyndale's version

that occurs	120	times	in Matthew
	78	"	" Mark
	161	"	" Luke
	144	"	" John
	503	"	" the four Gospels

In Wiclif's version

who occurs	8	times	in Matthew
	8	"	" Mark
	21	"	" Luke
	25	"	" John
	62	"	" the four Gospels

In Tyndale's version

who occurs	13	times	in Matthew
	10	"	" Mark
	21	"	" Luke
	30	"	" John
	74	"	" the four Gospels

In Wiclif's version

whose occurs	1		in Matthew
	0		" Mark
	3	times	" Luke
	5	"	" John
	9	"	" the four Gospels

In Tyndale's version

whose occurs	0		in Matthew
	1		" Mark
	5	times	" Luke
	5	"	" John
	11	"	" the four Gospels

The relatives *that*, *which* and *who* occur in Wiclif 1043 times, *that* 801 times or in about 76 per cent. of the cases, *which* 171 times or in 16.4 per cent., *who* or *whose* 71 times or in 6.8 per

cent. of all cases. In Tyndale's version we find a change, the same pronouns occur 1150 times; *which* has gained in frequency of use, occuring 562 times or in about 50 per cent. of all cases, *that* 503 times or in 44 per cent., *who* in 85 cases or in about 7.4 per cent.

To summarize: *that* occurs in Wiclif's version in 76 per cent. of all cases, but in the Tyndale version in only 44 per cent. of such cases, while *which*, appearing in but 16.4 per cent. of such cases in Wiclif, has risen to 50 per cent. in Tyndale, and soon becomes the leading relative.

In Tyndale's translation of 1526, a usage was established which was preserved with only limited exceptions in the King James version of 1611. As religion appeals to the strongest convictions of our nature, and is associated with glowing feeling, the fixed forms in which truth is conveyed in the Bible have stamped themselves upon human thought and expression. From the restricted use of *which* in 1200 it had in the fourteenth century, the period of Wiclif and Chaucer, attained a recognized currency, while 150 years later (1526) it divided almost equally the sovereignty with *that*.

The dominant use of *which* with nouns is a fact which we might have anticipated from the primitive meaning of *which*, hwi-líc or hwa-líc, *of what kind, how constituted*, like the Latin *qualis*. Substantives naturally possess character or quality, and the relative in referring to them means *of which kind*. *That* merely identifies and does not describe; similarly, *who* indicates usually an individual. Thus in Shakespeare, "I have known those *which* (*qualis*) walked in their sleep, *who* (equal to *and yet they*) died holily in their beds" (*Macbeth*, V, 1, 66). Quoted by Abbott, *Shakespearean Grammar*, page 182.

Which is uniformly employed with proper names: "And thou, Capernaum, *which* art exalted unto heaven" (Matthew 11 : 23); "Then cometh he to a city of Samaria, *which* is called Sychar" (John 4 : 5); "For he was father-in-law to Caiaphas, *which* was the high priest that same year" (John 18 : 13); "The same day came to him the Sadducees, *which* say that there is no such resurrection" (Matthew 22 : 23); occurring in such use 151 times, while *that* is similarly used but 5 times.

In Tyndale's version of 1526, *which* refers in the Gospels to a noun about 418 times, *that* to a noun 119 times, a total of 537 times, or in the proportion of 78 per cent. to 22 per cent. *Which*

refers to a noun denoting a place or thing 153 times, to a personal noun 265 times. *Which* refers to a personal, indefinite or demonstrative pronoun 144 times. *That* refers to a non-personal noun 77 times, to a noun denoting a person 42 times, or a total of 119. *That* refers to a pronoun 384 times. Out of 602 cases of the use of a simple relative referring to a pronoun, *that* is used in 64 per cent. of all cases, *which* in 23.5 per cent. of all cases.

The limited use of *who* in the Gospels in Tyndale's version is shown by the fact that out of about 1165 cases of the use of the simple relative, *who* is used only 55 times or a little more than in 5 per cent. of the cases.

The two translations of the Bible by Tyndale, 1526, and the King James version of 1611 present often kindred features in the use of words. The translators of the King James version adopted substantially the usage of the version of Tyndale. Nothing shows the dependence of the translators of the King James version upon Tyndale more than a comparison of the use and relative frequency of certain forms. We note a striking change which the language had undergone since the period of Wiclif. The relative pronoun *which* refers in the greatest number of cases to nouns, the relative pronoun *that*, in addition to its use with nouns, is used almost universally with personal and indefinite pronouns. The form of two petitions in the Lord's Prayer illustrate this usage, and have remained fixed in liturgical service to the present time: "Our Father *which* art in Heaven," "forgive us our sins, for we also forgive everyone *that* is indebted to us." The relative pronouns *which* and *that* occur in the four Gospels in the Tyndale version 1065 times. Of these, *that* is used 503 times, and *which* 562 times.

The use of the relatives *which* and *that* in the King James version does not differ greatly from the use of these pronouns in the version of Tyndale. In Tyndale, the relative pronoun *that* is used 32 times, where *which* is substituted in the King James version; *which* takes the place of *that* 4 times, and *which* is used 6 times instead of *who*, of the King James version, while in 60 cases an equivalent expression is used instead of a relative pronoun.

In Shakespeare, if we take the *Merchant of Venice* as representing fairly the plays, *that* is used 75 times, or in 83 per cent. of the restrictive clauses, while *which* is used in the same class of clauses 20 times, or in about 17 per cent.; *that* is used in co-ordinate clauses 11 times, or in 32 per cent., and *which* is used 23 times,

or in 68 per cent. of such cases. The usage which we have found in the King James version, and earlier in the Tyndale version, occurs also in Shakespeare. In the above play, *that* refers to personal nouns 15 times, or in about 88 per cent. of the cases, while *which* refers to personal pronouns but twice, or 12 per cent. *Who* refers to personal pronouns 26 times, to nouns 8 times, to animals personified once. In the entire play, *that* occurs 122 times, equal to 62.5 per cent., *which* 73 times, or 37.5 per cent. *Which* is used in restrictive clauses 20 times, in co-ordinate clauses 23 times.

The usage of Shakespeare is thus very flexible, showing greater variety and greater freedom, as we should expect, than occurs in the version of the Scriptures.

The relative pronoun was omitted in restrictive, but not in subordinate clauses. *Who* originally referred to things as well as to pronouns, and such use is familiar in Shakespeare. Thus, in the *Merchant of Venice*, the Prince of Morocco, in describing the three caskets, says: "The first of gold *who* (which) this inscription bears, *who* chooseth me shall gain what many men desire." "The second silver, *which* this promise carries, *who* chooses me shall get as much as he deserves."

A little later, *that* occurs, often with great uniformity, apparently to lend smoothness to the verse. "In the prologue of Fletcher's *Faithful Shepherdess* (1610), which was probably not written by Fletcher, *which* occurs, but *that* appears uniformly in the remaining acts of the play" (Morris).

A century later (1726), we find Swift using the relative *that* when the antecedent is a pronoun, thus following the usage in Tyndale and in the King James version of the Bible.

In the eighteenth century, there was a manifest effort on the part of certain writers to promote the use of *who* and *which* at the expense of *that*. We have in No. 78 of the *Spectator*, Steele's humorous plea in behalf of the restoration of *who* and *which* to their ancient rights: "We are descended of ancient families, and kept up our dignity and honor many years, till the jack-sprat *that* supplanted us. How often have we found ourselves slighted by the clergy in their pulpits and the lawyers at the bar. Nay, how often have we heard in one of the most polite and august assemblies in the universe, to our great mortification, these words, 'That that that noble lord urged'; which, if one of us had had justice done,

would have sounded nobler thus, 'That which that noble lord urged.' Senates themselves, the guardians of British liberty, have degraded us and preferred *that* to us; and yet no decree was ever given against us. In the very acts of Parliament, in which the utmost right should be done to everybody, word and thing, we find ourselves often either not used, or used one instead of another. In the first and best prayer children are taught they learn to misuse us. 'Our Father *which* art in Heaven' should be 'Our Father *who* art in Heaven'; and even a convocation, after long debates, refused to consent to an alteration of it. In our general confession we say, 'Spare thou them, O God, *which* confess their faults,' which ought to be '*who* confess their faults.' What hopes then have we of having justice done us, when the makers of our very prayers and laws, and the most learned in all faculties, seem to be in a confederacy against us, and our enemies themselves must be our Judges?"

Steele's view is specious, and is not based upon an accurate knowledge of the historical use of the relatives, or he may have had in mind a contemporary fashion in literature which he sought to counteract. If so, it is not clear against whom his shafts were directed.

In the Sir Roger de Coverley papers in the *Spectator*, written by Addison and Steele, the relatives *which* and *that* occur 531 times; of these, *which* is used 353 times, *that* 178 times. *Which* is used in restrictive clauses 179 times, or in 53 per cent. of all cases, *that* 161 times, or in 47 per cent. of all cases. *Which* refers to nouns 255 times, *that* to nouns 129 times. The influence of an antecedent modified by demonstrative or an indefinite pronoun, to which in certain instances the choice of the relative may be due, is shown by the fact that *which* refers to a noun so modified 83 times, equal to 76 per cent. of such cases; *that* refers similarly to a noun so modified in 26 cases, equal to 24 per cent. of such cases. *That* refers to a demonstrative or an indefinite pronoun 39 times, equal to 76½ per cent. of such cases, *which*, 12 times, equal to 23½ per cent. We see here a revival or perpetuation of the usage of the earlier centuries. In spite of the great influence ascribed, apparently erroneously, to Addison in re-establishing the use of *that*, he uses this relative only one-third as often as *which*.

In Macaulay's essay on Milton, the relative *which* occurs 191 times, *that* 7 times, total 198 times. *Which* refers to noun ante-

cedents 174 times, or in 99 per cent. of all cases; *that* refers to a noun antecedent but once. There is a striking use of *who* as a relative. This pronoun occurs in all 101 times; referring in 58 instances to a noun, and in 43 to a pronoun; to a personal pronoun 6 times, to a demonstrative or indefinite pronoun 37 times. This is the highest proportion obtained in the examination of any author. It shows a distinct mannerism, affecting noticeably the style of the historian. *That* as a relative occurs only 7 times. *Which* is used in restrictive clauses 198 times, or in 97 per cent. of all cases; *that* occurs in the same class of clauses 6 times, or in 3 per cent. of all cases. *Which* refers to an indefinite or demonstrative pronoun 10 times, or 71 per cent.; *that* 4 times, or 29 per cent. *What* is used 17 times. *Which* is used to introduce co-ordinate clauses 6 times, *that* in no instance. *Which* refers to an indefinite or demonstrative pronoun 13 times, or 81 per cent., *that* 3 times.

In the *Sartor Resartus* (1831) of Thomas Carlyle, the relatives *which* and *that* occur in all 393 times. *Which* is used in restrictive clauses 259 times, or in 66 per cent. of all cases, *that* 134 times, or in 34 per cent. of all cases. The relative in co-ordinate sentences is *which*, occurring 34 times, and is universally employed. *Which* is the relative employed with nouns, as in the King James version of the Bible, in about 243 instances, or in 90 per cent. of all cases. *What* is used as a relative 93 times, *that which* 4 times.

In Emerson's Essays, second series (1844), the relatives *which* and *that* occur 402 times; of these, *which* is used in restrictive clauses 344 times, or in about 86 per cent.; *that* is used in restrictive clauses 58 times, or in 14 per cent. of all cases. *Which* is used in co-ordinate sentences 27 times, or in all cases, *that* not at all. *What* is used 55 times, *that which* 21 times. *Which* nearly always relates to nouns, namely, in 330 out of 344 instances of its use.

Matthew Arnold, in his *Essays on Criticism* (1865), shows a uniform preference for *which* in both restrictive and co-ordinate clauses, greater variety and a more flexible adoption of one or the other relative. In four essays, namely, those on "Heinrich Heine," "A Guide to English Literature," "A French Critic on Goethe" and "George Sand," in 201 cases of the uses of the relatives *which* and *that*, these pronouns are used in restrictive clauses 188 times. *Which* is used in 186 instances, or in about

99 per cent. of all cases, *that* 2, or in 1 per cent.; *which* is used in co-ordinate clauses 12 times, *that* once. *What* is used as a relative 68 times in the same essays, *that which* 4 times. The use of *what* as a relative shows a steady and remarkable growth in frequency in later writers. Its use by Matthew Arnold in the above selections occurs 68 times, or in 25 per cent. of all cases of the use of a relative pronoun. *Which* is the common relative in co-ordinate clauses, being used in about 92 per cent. of all the cases.

The striking frequency of *which* in modern literature is shown in the writings of Mrs. Humphry Ward. The conclusions reached in our examination of the works of Macaulay and De Quincy are maintained, though not in as extreme a degree. Thus in *Robert Elsmere* (1888), Book 1, in about one-fourth of the volume, the relatives *that* and *which* occur 400 times; of these, *which* occurs 350 times, or about 87½ per cent., *that* 50 times, or 12½ per cent. Of relatives referring to noun antecedents *which* is used 341 times, to pronoun antecedents 9 times; *that* is used referring to a noun antecedent 41 times or 82 per cent., to a pronoun antecedent 9 times or 18 per cent. Mrs. Ward's use of these relatives is apparently confined to restrictive clauses.

Proverbs which have existed in the popular language for many centuries have preserved an archaic type of expression and are permanent representatives of primitive usage. Similarly children's rhymes, such as "The house *that* Jack built," which goes back to a mediæval Hebrew version in a hymn. In "This is the house that Jack built," "This is the malt *that* lay in the house *that* Jack built," we have the early use of the relative *that* in restrictive clauses; so also, in such proverbs as "Handsome is *that* handsome does," quoted from Goldsmith in the *Vicar of Wakefield*, chapter first; "He *that* will not when he may, when he will he shall have nay"; "There is none so blind as they *that* won't see"; "'Tis an ill dog *that* is not worth whistling for."

We thus see that the dominant relative *þē* of early English times was displaced by *that* in the thirteenth century, that *what* also appeared at that time in isolated cases in its relative use, while *who* and *whose* occur but seldom and then usually in direct and indirect questions. At the close of the fourteenth century, *that* was used in Wiclif's translations of the Gospels in 76 per cent. of all cases of the use of the relative, *which* in 16 per cent. of such cases.

One hundred and fifty years later, in 1526, *that* occurs as a rela-

tive in the Tyndale version in only 44 per cent. of all cases, while *which* has risen from 16 per cent. in Wiclif to 50 per cent. in Tyndale. *Which* was confined largely to nouns and *that* to pronouns. In the eighteenth century, *which* declines in use in the classical English of Addison and Steele, while *that* gains slightly in frequency. A more marked change is manifest in the nineteenth century in the English of Macaulay, where *which* refers to a noun in 99 per cent. of all cases of its use as a relative, constituting a marked feature of his style. In Matthew Arnold, this proportion is preserved; also, though in a less degree, in the writings of Mrs. Humphry Ward. The present tendency is to subordinate the use of *that*, perhaps in part due to its use as a declarative conjunction, while *who* has gained in frequency of use and refers mainly to personal nouns.

Cornell University, Ithaca, April 9, 1904.

OPISTHENOGENESIS, OR THE DEVELOPMENT OF SEGMENTS, MEDIAN TUBERCLES AND MARKINGS *A TERGO*.

BY ALPHEUS S. PACKARD, LL.D.

(*Received June 15, 1904.*)

Weismann, in his suggestive *Studies in the Theory of Descent* (1876), was the first to discuss the origin of the markings of caterpillars, and to show that in *Deilephila hippophaës* the ring-like spots of the larva "first originated on the segment bearing the caudal horn, and were then gradually transferred as secondary spots to the preceding segments" (Vol. 1, p. 277).

Afterwards (1881–1890), Eimer[1] showed that in the European wall-lizard "a series of markings pass in succession over the body from behind forwards, just as one wave follows another, and the anterior ones vanish while new ones appear behind." He speaks

[1] "Untersuchungen ueber das Variiren der Mauereidechse," *Archiv f. Naturg.*, 1881; "Ueber die Zeichnung der Thiere," *Zool. Anzeiger*, 1882, 1883, 1884; *Organic Evolution*, London, 1890

of this mode of origin of the markings as the "law of wave-like evolution, or law of undulation." In confirmation of this process or law he cites the conclusions of Würtenberger,[1] who had long before (1873) observed that in ammonites all structural changes show themselves first on the last (the outer) whorl of the shell, such a change in the following generations being pushed farther and farther towards the beginning of the spiral, until it prevails in the greater number of the whorls."

Cope, in his *Primary Factors of Organic Evolution* (1896), also shows that in the lizards *Cnemidophorus tesselatus* and *gularis* the breaking up of the striped coloration into transverse spots begins first at the sacral and lumbar regions: "The confluence of the spots appears there first."

We may cite some examples of this law of growth *a tergo*, or opisthenogenesis, as it might be called, which have fallen under our own observation.[2]

In *Dasylophia anguina*, as shown by the figures in Plate XXI of my monograph of the bombycine Moths, Pt. 1, it will be observed that in stages III, IV and the last stage the dark longitudinal lines become on the eighth to tenth abdominal segments broken up into separate isolated dark spots. In the larva, before the second molt, there are no spots on the ninth and tenth segments. In stage III, however, *i.e.*, after the second change of skin, as stated in my monograph (p. 175), four black spots now appear on the front part of the suranal plate. In the last stage, the reddish spots on the eighth abdominal segment which are detached from the lateral lines of stages I and II, now become specialized into the two black comma-like spots, with a linear spot above and beneath; the two, sometimes divided into four, black spots arise on the suranal plate.

It thus appears that in the ontogeny of this species the process of breaking up or origin of the spots from the longitudinal lines takes place on the last three segments of the body.

In *Symmerista albifrons* the same phenomenon occurs. In stage I, as stated in my monograph (p. 180), on each side of the ninth segment is a large black comma-shaped spot, the point directed forward and downward, while behind there is a median black dot.

[1] *A New Contribution to the Zoological Proof of the Darwinian Theory*, Ausland, 1873, Nos. 1, 2, and *Studies on the History of the Descent of the Ammonites*, Leipzig, 1880 (in German).

[2] *Proc. Amer. Asso. Advancement Science*, Boston Meeting, 1898, pp. 368–9.

After the first molt there arises behind the dorsal hump two, instead of one, median black spots, and two black spots are added on the side of the body near the base of the anal legs, *i.e.*, two each on the ninth and last segments.

After the second casting of the skin, the marking of the three last abdominal segments becomes specialized; what on the body in front are parallel black and red lines being in this region now represented by separate spots. Thus as regards the marking, the anterior part of the body remains ornamented with the primitive parallel lines; while the process becomes on the three hinder segments accelerated or specialized. It thus appears that the more advanced or ontogenetically later style of ornamentation originates at the end of the body.

A parallel process takes place with the formation of the caudal horn or hump. Thus in Symmerista, Dasylophia and other horned Notodontidæ and members of other groups, the eighth abdominal segment is the theatre of the process of fusion of the two dorsal tubercles of the first larval stage into a single tubercle or horn; so that this segment appears to be the theatre of a process of specialization which does not take place on any other segments of the body.

When in other genera it does take place and there is a specialized single tubercle on the first abdominal segment, as in Notodonta, Nerice and more especially in Hyparpax and Schizura, the process of fusion of two tubercles into a single specialized one, as on abdominal segments 1, proceeds from behind forward, as it were in waves of translation of the specialized growth-force from behind forwards.

This may clearly be seen in the figures on Plate XXIV, showing the development of the single hump in *Hyparpax aurora*. In Fig. 1, the dorsal tubercles *i* in stage I are all separated; in Fig. 2, those on the eighth abdominal segment have all begun to unite at their bases before they have on the first abdominal segment; they seem to be a little behind at first, though later on the hump on the first segment becomes higher and larger than the caudal horn.

If there were any doubt as to the relative period when the tubercles become fused in Hyparpax, in *Schizura leptinoides* (Pl. XXVI) it is very clearly shown by Fig. 1 that the fusion of the two tubercles forming the caudal hump as we will call it, *i.e.*, that on the eighth abdominal segment, has taken place before any signs

of such fusion have appeared in the pair on any of the segments in front.

When the ontogeny of *Nerice bidentata* is worked out, it will be a matter of much interest to observe whether the dorsal humps are formed from behind forward, or whether they appear simultaneously, and thus form an apparent exception to the law of transfer of growth-force from behind forwards.

In this connection it might be observed that in the larva of *Schizura unicornis*, in which there is the very unusual occurrence of a pair of short thick spines on the vertex of the head (Pl. XXVIII, Fig. 2, 2a, 2b), these spines do not appear in stage I and not until after the first molt. These spines persist through stages II and III, but after this disappear, not being present in the two last stages. Thus the growth-force resulting in the development of the armature of stage I does not reach the head until after the first molt, and then does not persist throughout larval life.

In the ontogeny of the Notodontian family, as well as that of Ceratocampidæ and Saturniidæ, the process of fusion of the two dorsal tubercles always first begins on the eighth abdominal segment.

Opisthenogenesis, as regards the markings, appears to be of a piece, or somehow connected, with the opisthenogenetic origin in post embryonic development of new segments. In the cestodes and in annelid worms, multiplication of segments occurs between the head-region and the extreme end of the body. Thus in Polygordius, as stated by Balfour (*A Treatise on Comparative Embryology*, 1880, I, pp. 271, 272), the conversion of the larva into the adult takes place "by the intercalation of a segmented region between a large mouth-bearing portion of the primitive body and a small anus-bearing portion."

This region in the larval or early stages of worms and more primitive arthropods is the "budding zone" of embryologists. While at the outset, in the beginning of embryonic life, the head-region is the first to be formed and the trunk-segments arise later, as in the trochosphere of worms and the protaspis of trilobites and of merostomes, a third portion, arising from the budding zone or seat of rapid cell-formation, appears to be a secondary or inherited region, due to the post-embryonic acquisition of new characters (certain trunk-segments and their appendages) in many segmented or polymerous animals, *i.e.*, those which have passed beyond the trochozoön stage or type.

Prof. E. B. Wilson[1] has clearly stated the nature, now so well known, of the growth-processes involved in the interpolation at the growing point or budding-zone of new segments. In Polygordius, after the trochosphere has been formed and when it is about to enter on the adult stages, the segments are formed successively, those in front being the oldest, "while new segments are continually in process of formation, one after another, at the growing point." This, he says, is "a typical case of apical or unipolar growth." It is what we would call opisthenogenetic growth.

Professor Whitman[2] has shown that in the leech the internal tissues (mesoblast) of the budding zone are arranged in two widely separated lateral bands which, to quote Wilson's exposition, "as the trunk grows older, widen out and grow together along the median line, ultimately giving rise to muscles, blood-vessels, excretory organs, reproductive organs, etc." Now if this is the case with the more important tissues, why in caterpillars as well as in lizards may not this opisthenogenetic mode of growth also involve the arrangement and distribution of the pigment-masses of the integument?

Without entering into the mode of development of the germ-bands, which are behind completely separated, gradually becoming united in front, resulting in their union or concrescence, we would make the suggestion that this phenomenon may be the initial cause or at least in some way connected with the breaking up of the longitudinal stripes of the body, and their transformation into spots at or near the budding zone of their polymerous or polypodous (Peripatus-like) ancestors.

In the trilobites, Limulus and Diplopods, the new segments after embryonic life are interpolated between the penultimate and anal or last segment of the body, and it is from this region in certain Lepidopterous larvæ that the transformation of longitudinal stripes into spots takes place. The question next arises whether there is any connection between the opisthenogenetic origin of the markings of lizards and that of caterpillars. The fact, now well established by embryologists, that the phenomena of concrescence occurs not only

[1] *Some Problems of Annelid Morphology.* Biological lectures delivered at the Marine Biological Laboratory at Woods Holl, 1891, p. 61. See also A. D. Mead, "The Early Development of Marine Annelids," *Journal of Morphology*, XIII, May, 1897, pp. 227–326.

[2] "The Embryology of Clepsine," *Journ. Micr. Sc.*, XVIII, 1878; *Journal of Morphology*, Boston, 1887. I am indebted to Prof. A. D. Mead for calling my attention to the concrescence process in this connection.

in fishes but in Amphibia and reptiles, would suggest that the cause of the transformation of longitudinal stripes into spots on the lumbar and sacral regions of lizards is the result of the same specializing growth-force. It may perhaps be regarded as a surviving remnant of the segment-forming force, which has affected the pigment bands in a manner identical in the vertebrates and insects. This transformation of stripes into spots, and the fusion of two dorsal tubercles into a median one, may be, then, the sign of some latent or surviving amount of force concerned in the origin and formation of segments, which crops cut in the larval stages of insects and in young lizards, resulting in this opisthenogenetic mode of origin of spots from bands.

ORTHIC CURVES; OR, ALGEBRAIC CURVES WHICH SATISFY LAPLACE'S EQUATION IN TWO DIMENSIONS.

BY CHARLES EDWARD BROOKS, A.B.

(*Read May 20, 1904.*)

I propose a study of the metrical properties of algebraic plane curves which are apolar, or, as it is sometimes called, harmonic, with the absolute conic at infinity. If we disregard the right line, the simplest orthic curve is the equilateral (conic) hyperbola, and the name equilateral hyperbola is sometimes extended to orthic curves of higher order. Doctor Holzmüller,[1] who devotes a section to curves of this kind, calls them hyperbolas; and M. Lucas[2] calls them "stelloides." M. Paul Serret, in a series of three papers in *Comptes Rendus*,[3] uses the word "équilatère" for a curve with

[1] *Einführung in die Theorie der Isogonalen Verwandschaften und der Conformen Abbildungen*, Gustav Holzmüller, Leipzig, 1882, p. 202. . . .

[2] "Géométrie des Polynomes," Felix Lucas, *Journal de l'Ecole Polytechnique*, 1879, t. XXVIII.

[3] *Comptes Rendus*, 1895, t. 121. Sur les hyperboles équilatères d'ordre quelconque, p. 340.

Sur les faisceaux regulieres et les équilatères d'ordre n. p. 372.

Sur les équilatères comprises dans les equations

$$o = \Sigma_1^{2n-2} l, T_1^n = H_n,$$

$$o = \Sigma_1^{2n-1} l, T_1^n = H_n + \lambda H'.$$

p. 438.

asymptotes concurrent and parallel to the sides of a regular polygon. It seems advisable to follow M. Serret's usage, and to denote such a curve by the name equilateral, using another term to express apolarity with the absolute. For this purpose I have adopted the word orthic.

If we use Cartesian coördinates, a curve

$$U(XY) = 0,$$

is apolar with the absolute conic,

$$\xi^2 + \eta^2 = 0,$$

if

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0.$$

In other words, an orthic curve is one which satisfies Laplace's equation in two dimensions.

Part One—The Orthic Cubic Curve.

I. *The Condition that a Curve be Orthic.*

In the analysis which may be required, I shall employ conjugate coördinates, x, $\bar{x}$, which may be defined as follows: If X and Y are rectangular Cartesian coördinates of any point, the conjugate coördinates of that point are

$$x = X + iY, \; \bar{x} = X - iY,$$

when the origin is retained, and the axis of X is chosen as the axis of reals, or base line. It is sometimes convenient to think of x as the vector from the origin to the point, and of $\bar{x}$ as the reflection of this vector in the base line. If x, $\bar{x}$ is a real point of the plane, not on the base line, $x - \bar{x} = 0$, x and $\bar{x}$ are conjugate complex numbers. Since if one of its coördinates is known the other is immediately obtainable, we shall, as a rule, name a point by giving only one of its coördinates. It is convenient to reserve the letters t and τ for points on the unit circle,

$$x\bar{x} = 1.$$

Now, Laplace's equation,

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0,$$

when applied to a function of x and $\bar{x}$, becomes

$$\frac{\partial^2 U(x\bar{x})}{\partial x \partial \bar{x}} = 0.$$

It follows that:

The necessary and sufficient condition that a curve be orthic is that its equation in conjugate coördinates contain no product-term.

II. *Kinematical Definition of the Orthic Curve.*

Let us now proceed to the study of the orthic curve of the third order. I shall obtain the equation of an orthic cubic in a way which will suggest immediately a method for the construction of points on the curve.

The path of a point which moves in such a way that it preserves a constant orientation from three fixed points is an orthic cubic curve.

If x is the moving point, and the three fixed points are α, β, γ, then the sum of the amplitudes of the strokes which connect x with α, β, γ, must remain constant. That is, we must have

$$(x-\alpha)(x-\beta)(x-\gamma) = \rho\tau_1.$$

If the curve is to be real, the conjugate relation,

$$(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\bar{\gamma}) = \rho\tau_1^{-1},$$

must hold simultaneously.

The equation of the curve is obtained by eliminating the parameter ρ between these. It is

$$x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$$
$$= \tau_1^2\{\bar{x}^3 - (\bar{\alpha}+\bar{\beta}+\bar{\gamma})\bar{x}^2 + (\overline{\alpha\beta}+\overline{\beta\gamma}+\overline{\gamma\alpha})\bar{x} - \overline{\alpha\beta\gamma}\}.$$

This is the most general equation of the third degree which we can have without introducing the product. As a consequence it represents a perfectly general orthic cubic.

If we transform to

$$x = \tfrac{1}{3}(\alpha+\beta+\gamma),$$

the centroid of $\alpha\beta\gamma$, as a new origin, and so choose the base line that τ_1^3 is real, the equation takes the form

$$x^3 + a_0 x + a_1 + \bar{a}_0 \bar{x} + \bar{x}^3 = 0.$$

The equation of any orthic cubic can be brought to this form. The three points, α, β and γ, are on the curve, and form what it is convenient to call a triad of the curve.

III. *The Orthic Curve is an Equilateral Curve.*

Consider the orthic cubic,

$$x^3 - s_1 x^2 + s_2 x - s_3 = \tau_1^2 (\bar{x}^3 - \bar{s}_1 \bar{x}^2 + \bar{s}_2 \bar{x} - \bar{s}_3),$$

where the s's are the elementary symmetrical functions of α, β γ. The approximation at infinity,

$$(x - \tfrac{1}{3} s_1)^3 - \tau_1^2 (\bar{x} - \tfrac{1}{3} \bar{s}_1)^3 = 0,$$

makes both the square and the cube terms vanish, and therefore represents the asymptotes. The factors of this are:

$$x - \tfrac{1}{3} s_1 - \sqrt[3]{\tau_1^2} (\bar{x} - \tfrac{1}{3} \bar{s}_1) = 0,$$

$$x - \tfrac{1}{3} s_1 - \omega \sqrt[3]{\tau_1^2} (\bar{x} - \tfrac{1}{3} \bar{s}_1) = 0,$$

$$x - \tfrac{1}{3} s_1 - \omega^2 \sqrt[3]{\tau_1^2} (\bar{x} - \tfrac{1}{3} \bar{s}_1) = 0.$$

where $\omega^3 = 1$.

These three lines meet at the point

$$x = \tfrac{1}{3} (\alpha + \beta + \gamma)$$

which we may call the centre of the curve. We notice that:

The centre of the orthic cubic is the centroid of the triad.

The clinants of the asymptotes are $\tau_1^{\frac{1}{3}}$, $\omega \tau_1^{\frac{1}{3}}$, $\omega^2 \tau_1^{\frac{1}{3}}$. They differ only by the constant factor ω. Now we know that multiplying the clinant of a line by ω is equivalent to turning the line through an angle $\frac{2\pi}{3}$. A rotation $\frac{2\pi}{3}$ about the centre sends each asymptote into another. It follows that the asymptotes of an orthic cubic are concurrent and parallel to the sides of a regular triangle. M. Serret[1] calls such a figure of equally inclined lines which meet in a point a regular pencil, and a curve with asymptotes forming a regular pencil he calls an "*équilatère.*"

[1] *Comptes Rendus*, Sur les hyperboles équilatères d'ordre quelconque. 1895, t. 121, p. 340.

Now any cubic curve, the asymptotes of which form a regular pencil, can be brought to the form :

$$x^3 + a_0 x + a_1 + \overline{a_0 x} + \bar{x}^3 = 0,$$

in which we recognize it as orthic. It follows that :

The orthic cubic and the equilateral of order three are identical.

The relation

$$(x - \alpha)(x - \beta)(x - \gamma) = \rho\tau_1 = z$$

may be regarded as mapping a line through the origin in the z plane,

$$z - \tau_1^3 \bar{z} = 0$$

into the orthic cubic. We are thus able to identify the latter with the curves discussed by Holzmüller[1] and by Lucas.[2]

IV. *Construction of Points of an Orthic Cubic.*

A figure of the orthic cubic may be obtained without great difficulty by constructing points of the curve. In order to show how this may be done, it is necessary to prove the following lemma :

Elements of the pencil of equilateral (orthic) hyperbolas, of which the stroke $\beta\gamma$ is a diameter, intersect corresponding elements of the pencil of lines through α on an orthic cubic of which $\alpha\beta\gamma$ is a triad.

For the line through α,

$$(x - \alpha) = \rho\tau',$$

and the equilateral hyperbola on $\beta\gamma$ as a diameter,

$$(x - \beta)(x - \gamma) = \rho\tau'',$$

intersect on the orthic cubic

$$(x - \alpha)(x - \beta)(x - \gamma) = \rho\tau_1$$

[1] Holzmüller, *Conformen Abbildungen*, p. 205.

[2] Lucas, Géométrie des Polynomes, *Journal de l'Ecole Polytechnique*, t. XXVIII, p. 23.

if

$$\tau'\tau'' = \tau_1.$$

If the two pencils are given, it is only necessary to pair off lines and curves according to the relation

$$\tau'\tau'' = \tau_1,$$

and to mark intersections. These will be points of the curve.

A very simple instrument for drawing the equilateral hyperbolas required in the construction is made in the following way: Two toothed wheels of equal diameters are attached beneath the drawing

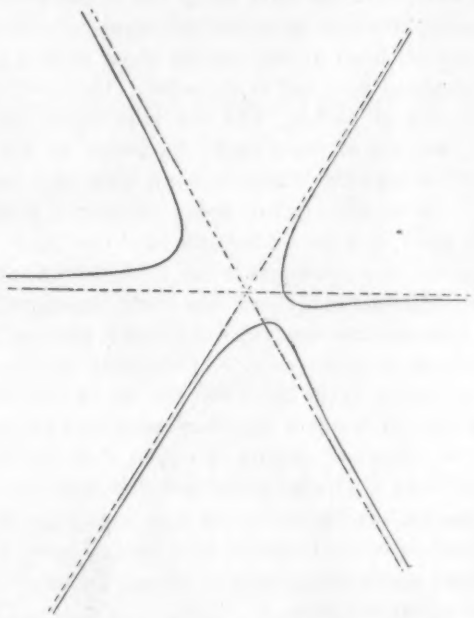

Figure 1. A unipartite orthic cubic which has three real inflections, one of which is at infinity.

board in such a way that their teeth engage. The axles are perpendicular to the board and come through it at β and γ. The axles, which turn with the wheels, carry long hands or pointers which sweep over the board. On account of the cogs, the wheels can turn only through equal and opposite angles. As a consequence, x, the

intersection of the hands, has a constant orientation from β and γ, and in fact generates the orthic curve of the second order given by

$$(x-\beta)(x-\gamma)=\rho\tau',$$

which is the hyperbola required.

V. *Mechanical Generation of an Orthic Cubic.*

A mechanism which will actually draw an orthic cubic is very much to be desired. One might be made in some such way as the following: Suppose three hands like those described above (IV) to be pivoted at α, β and γ. Let them be held together in such a way that, while each is free to move along the others, they must always meet in a point, which is to be the tracing point. Each hand is to receive its motion from a cord wound about a bobbin on its axle. The bobbins are to be equal in diameter. The cords pass through conveniently placed pulleys, and are kept tight and vertical by small equal weights at their ends. Consider, to fix ideas, those three weights which by their descent give the hands positive rotation. If, now, the tracing point be moved along an orthic cubic which has α, β, γ for a triad, the total turning of the bobbins will be zero, and as a consequence the total descent of the weights will be zero. Conversely, if we can move these vertically and in such a way that the total descent will be zero, the tracing point can move only along an orthic cubic. This result will be obtained if the centre of gravity of the three weights can be kept fixed. It will not do, however, to connect the three weights by a rigid triangle pivoted at its centre of gravity, for then they will not move vertically. But since a parallel projection does not alter the centroid of a set of points, the desired result will be attained if the weights are constrained to vertical motion by guides of some kind, and are kept in a plane which always passes through the centre of gravity of one position of the weights.

VI. *The Orthic Cubic through Six Points of a Circle.*

Consider the general orthic cubic given by

$$x^3-a_0x^2+a_1x-a_2+a_3\bar{x}-a_4\bar{x}^2+a_5\bar{x}^3=0.$$

It cuts the unit circle,

$$x\bar{x}=1,$$

in six points, the roots of

$$x^6 - a_0x^5 + a_1x^4 - a_2x^3 + a_3x^2 - a_4x + a_5 = 0.$$

If we want the cubic to meet the circle in six given points, say τ_1, τ_2, . . . τ_6, then this equation must be identical with

$$x^6 - s_1x^5 + s_2x^4 - s_3x^3 + s_4x^2 - s_5x + s_6 = 0,$$

in which the s's stand for the elementary symmetrical combinations of the six τ's. This requires

$$a_0 = s_1,\ a_1 = s_2,\ a_2 = s_3,$$
$$a_3 = s_4,\ a_4 = s_5,\ a_5 = s_6$$

The coefficients of the cubic equation are then precisely determined, with the result that:

But one orthic cubic can be constructed through any six points of a circle.

It remains for us to show that one such curve can always be drawn: that is, that the equation

$$x^3 - s_1x^2 + s_2x - s_3 + s_4\bar{x} - s_5\bar{x}^2 + s_6\bar{x}^3 = 0$$

always represents a real curve. If we so choose the base line that $s_6 = 1$ then we have

$$\bar{s}_1 = s_{6-1}\, s_6^{-1} = s_{6-1},$$

and the equation takes the form

$$x^3 - s_1x^2 + s_2x - s_3 + \overline{s_2x} - \overline{s_1x^2} + \bar{x}^3 = 0,$$

which is, obviously, self-conjugate, and is therefore satisfied by the coördinates of real points. As a result:

An orthic cubic can always be drawn through six points of a circle. It is then determined uniquely.

VII. *The Intersections of an Orthic Cubic with a Circle.*

When the orthic cubic is referred to the six points in which it cuts the unit circle, the equations of the asymptotes take the form

$$x - \tfrac{1}{3}\, s_1 = (-s_6)^{\frac{1}{3}}\omega^i\,(\bar{x} - \tfrac{1}{3}\, s_5 s_6^{-1}).$$
$$i = 0,\ 1,\ 2.$$

These three lines meet at

$$x = \tfrac{1}{3} s_1,$$

the centre. This point, the origin, and the point which is the centroid of the six points on the circle lie on a line; and the latter point is midway between the other two. This leads to the interesting fact that:

The centroid of the six points in which any circle meets an orthic cubic bisects the stroke from the centre of the curve to the centre of that circle.

VIII. *Triads of the Curve.*

We spoke of the three points α, β, γ, which have the same orientation from every point of the curve, as a triad of the curve. Let us see how many such triads there are, and how they are arranged. The relation

$$(x - \alpha)(x - \beta)(x - \gamma) = z$$

may be regarded as establishing a correspondence between points x in one plane and points z in another plane, in such a way that if z describe a line ξ through the origin, the point x generates an orthic cubic on $\alpha\beta\gamma$ as a triad. To every position of z on the director line ξ there correspond three points in the x-plane. I shall show that each such set of three points is a triad. Write

$$F(x) = (x - \alpha)(x - \beta)(x - \gamma).$$

Then, if x_1, x_2, x_3, are the three points which correspond to z,

$$F(x) - z = (x - x_1)(x - x_2)(x - x_3).$$

And also

$$F(x) - z' = (x - x_1')(x - x_2')(x - x_3').$$

Now this relation is satisfied by x_1, or x_2, or x_3.

$$F(x_1) - z' = (x_1 - x_1')(x_1 - x_2')(x_1 - x_3') = z - z'.$$

Since $z - z'$ is a point of the director line, it follows that the three points x_1', x_2', x_3', which correspond to any point z' of the director line, have the same orientation from every point of the curve. We conclue that:

To every point of the director line corresponds a triad; all the

points of the curve have the same orientation from any triad, and all the triads of the curve have the same orientation from any point of the curve.

IX. *The System of Confocal Ellipses Connected with the Triads.*

We seek the points of a triad which correspond to a given point z.[1] The map equation can be brought to the form

$$x^3 - 3x = 2z$$

by choosing the centre of the curve as a new origin and making a suitable choice of the unit stroke. We see at once that the sum of the x's for a given z is zero. In other words: *The centroid of any triad is the centre of the cubic.*

Making use of the method known as Cardan's solution, put

$$x = \mu t + v,$$

where μ is real. Then

$$x^3 - 3x = 2z$$

becomes

$$\mu^3 t^3 + v^3 + 3\mu^3 t^2 v + 3\mu t v^2 - 3(\mu t + v) = 2z$$

And we have as two relations between v and μt,

$$2z = \mu^3 t^3 + v^3,$$

and

$$(\mu t + v)(\mu t v - 1) = 0.$$

When z is zero, the values of x are $\pm\sqrt{3}$ and 0; and when z is not zero, we must have

$$v = \frac{1}{\mu t}.$$

This leads to the expression of x and z in terms of μt as follows:

$$x = \mu t + \frac{1}{\mu t},$$

$$2z = \mu^3 t^3 + \frac{1}{\mu^3 t^3}.$$

Now if we assign any value to μ, and let t run around the unit circle, x describes an ellipse with foci at $x = +2$ and $x = -2$.

[1] Harkness and Morley, *A Treatise on the Theory of Functions*, p. 39.

But at the same time, z also describes an ellipse with its foci at $z = +1$ and $z = -1$. These two ellipses are related in such a way that a point z on one of them is correlated by the equation

$$x^3 - 3x = 2z$$

with three points on the other. Now the foci of both these ellipses are independent of the particular value of μ selected; it follows that if we assign successive values to μ, we shall obtain in each plane a system of confocal ellipses of such a sort that the equation

$$x^3 - 3x = 2z$$

establishes a one to one correspondence between them. In each plane the origin is the centre of all the ellipses. Applying this scheme to the case in hand, we see that a triad must be inscribed in one of the ellipses in the x-plane. But the centroid of the triad is the centre of the ellipse; so the ellipse must be the circumscribed ellipse of least area of that triad. We may say, then, that:

The triads of the orthic cubic are cut out on the curve by a particular system of confocal ellipses, and each ellipse is the circumscribed ellipse of least area of the triad on it.

X. *The Riemann Surface for an Orthic Cubic.*

If we examine the equation

$$x^3 - 3x = 2z$$

for equal roots, we find that the double points of the x-plane are at $x = +1$ and at $x = -1$. These values of x correspond to the branch points in the z-plane, $z = +1$ and $z = -1$.

Let us for a moment replace the z-plane by a three-sheeted Riemann surface. All three sheets must hang together at infinity, and two sheets at each of the branch points. Let the first and second sheets be connected by a bridge along the base line from $+1$ to infinity, and the second and third sheets be similarly connected by a bridge along the real axis from -1 to infinity.

Select on this surface any large ellipse with foci at the branch points, and any line as a director line. Now consider the contour obtained by starting from a point of this inside the ellipse, going thence along the line to meet the ellipse, along an arc of the ellipse to meet the line, and then along the line to the point of departure.

We can choose this path in such a way that one of the following three cases must arise:

(1) *The contour passes through a branch point.*

(2) *The contour surrounds two branch points.*

(3) *The contour surrounds no branch point.*

In case (1) we know that the cubic must have a node. In the second case, by going three times around we can pass continuously through every sheet of the Riemann surface and therefore through every value of x. Or, thinking again of the x-plane, we have a unicursal boundary. Now it happens that the ellipse we choose maps into one and not three ellipses on the x-plane. If we imagine this to expand indefinitely we shall have to consider the boundary as our orthic cubic. It follows at once that:

The orthic cubic which corresponds to a line which does not pass between the branch points is unipartite.

If the contour includes one branch point, and therefore crosses one bridge of the Riemann surface, we must go along two unconnected curves to reach all the values of x. When these two curves are spread on the x-plane they lead at once to the conclusion that:

The orthic cubic which corresponds to a line which passes between the branch points is a bipartite curve.

XI. *Triads in Special Cases.*

Let us turn our attention again to the two planes connected by the relation

$$x^3 - 3x = 2z$$

We notice that while the ellipses in the z-plane have their foci at the branch points, the foci of the corresponding system of ellipses are not the double points of the x-plane, but are the points $x = +2$ and $x = -2$, each of which, with one of the double points counted twice, forms a triad.

As a rule there are two triads of the curve on each ellipse, corresponding to the two points in which the director line cuts an ellipse of the system in the z-plane. But unless the line go between the branch points it will be tangent to one ellipse, consequently two triads will coincide, and the cubic will be tangent at three places to one of the ellipses of the system. No part of the cubic can be inside of that ellipse.

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When μ is 1, the two ellipses degenerate into two segments,

$$x = t + t^{-1} \text{ or } \overline{2, -2},$$

$$2z = t^2 + t^{-2} \text{ or } \overline{1, -1}.$$

If the line pass between the branch points, and so cut the segment $\overline{1, -1}$, two triads again coincide, but in this case the three points lie on a line, and we do not have the triply tangent ellipse.

When the line ξ cuts the axis of imaginaries,

$$z + \bar{z} = 0,$$

we have

$$z = \rho\, \epsilon^{\frac{\pi i}{2}},$$

and

$$t^2 = \rho'\, \epsilon^{\frac{\pi i}{2}}.$$

It follows that $am\, t = \frac{\pi}{4}$, and so ωt is the reflection of t in the axis of imaginaries and $\omega^2 t$ is a pure imaginary. Then, since we know that

$$x_i = \omega^i \mu t + \frac{1}{\omega^i \mu t},$$

$$i = 1, 2, 3,$$

we see that x_1 is the reflection of x_3 in the line $x + \bar{x} = 0$, and that x_2 is on that line. It follows that the triangle $x_1 x_2 x_3$ is isosceles and that its base $x_1 x_2$ is parallel to the real axis. There is again an isosceles triangle when t^2 is real. This triangle has its vertex on the axis of reals and its base perpendicular to that axis. From the discriminant of the quadratic in $\mu^2 t^2$,

$$z^2 - 4,$$

we see that t^2 is real when $z > \pm 1$. In other words, if the director line ξ cut the axis of reals, but not between the branch points, we have such an isosceles triangle.

From the above considerations, we see that if the director line is either of the axes

$$x + \bar{x} = 0,\ x - \bar{x} = 0,$$

then one branch of the orthic cubic must be a right line; the re-

maining portion of the curve must then be an ordinary hyperbola, and the inclination of its asymptotes must be either $\frac{\pi}{3}$ or $\frac{2}{3}\pi$. The first value refers to the case when the director line is the axis of imaginaries; and the last, to the case when it is the axis of reals.

XII. *The Intersections of the Circumscribed Circle of a Triad with the Cubic.*

Suppose we put a circle through the points of a triad, and ask, Where are the remaining three points in which it cuts the cubic? For convenience, let three points of the unit circle be taken as a triad. The cubic is then

$$(x - t_1)(x - t_2)(x - t_3) = \tau_1^3 (\bar{x} - t_1^{-1})(\bar{x} - t_2^{-1})(\bar{x} - t_3^{-1}).$$

On eliminating $\bar{x}$ from this and the equation of the circle we obtain

$$(x - t_1)(x - t_2)(x - t_3) = \frac{\tau_1^2 (t_1 - x)(t_2 - x)(t_3 - x)}{t_1 t_2 t_3 x^3},$$

or

$$x^3 = \frac{-\tau_1^2}{t_1 t_2 t_3},$$

as the equation of the three points sought. The roots of this,

$$x_1 = \kappa, \quad x_2 = \omega\kappa, \quad x_3 = \omega^2\kappa,$$

are the coördinates of the vertices of an equilateral triangle. As there is no restriction in taking the triad on the unit circle, we have the following theorem:

If a circle cut an orthic cubic in a triad, then the two curves have three other intersections, which form an equilateral triangle.

XIII. *The Pencil of Orthic Cubics Which Have a Triad in Common.*

We have seen that the relation

$$(x - \alpha)(x - \beta)(x - \gamma) = z$$

maps a line through the origin into an orthic cubic of which $\alpha\beta\gamma$ is a triad. It must then map all the lines through the origin into a single infinity of orthic curves[1] which have the common triad $\alpha\beta\gamma$.

[1] Felix Lucas, *Journal de l'Ecole Polytechnique*, t. XVIII, p. 21.

If we regard τ as a parameter, we may say that

$$(x-\alpha)(x-\beta)(x-\gamma)=\tau(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\gamma)$$

is the equation of the pencil of orthic cubics which have the triad $\alpha\beta\gamma$. It will be convenient to give a pencil of this sort some name; let us refer to it as a *central pencil*, noting for our justification that the centroid of the triad is the centre of every curve of the pencil.

If there were any real point other than α, β, or γ, on two curves of this pencil, it would map into a real point of the z-plane, not the origin, which would be on two of the lines through the origin. As this is manifestly impossible, it follows that:

Two orthic cubics which have a triad in common have no other real intersection.

Now we know that two cubic curves intersect in nine points, and that if the curves given by the equation

$$(x-\alpha)(x-\beta)(x-\gamma)=\tau(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\bar{\gamma})$$

really constitute a pencil, there must be six imaginary points whose coördinates satisfy this equation whatever the value of τ. Let us form the following table of coördinates. The real intersections are

$$x_1=\alpha,\ \bar{x}_1=\bar{\alpha},$$

$$x_2=\beta,\ \bar{x}_2=\bar{\beta},$$

$$x_3=\gamma,\ \bar{x}_3=\bar{\gamma}.$$

It is evident that each of the following points:

$$x_4=\alpha,\ \bar{x}_4=\bar{\beta},$$

$$x_5=\alpha,\ \bar{x}_5=\bar{\gamma},$$

$$x_6=\beta,\ \bar{x}_6=\bar{\alpha},$$

$$x_7=\beta,\ \bar{x}_7=\bar{\gamma},$$

$$x_8=\gamma,\ \bar{x}_8=\bar{\alpha},$$

$$x_9=\gamma,\ \bar{x}_9=\bar{\beta},$$

satisfies the equation, independently of τ. These points, the six imaginary intersections of the pencil, are the antipoints[1] obtained

[1] Cayley, *Collected Mathematical Papers*, Volume VI, p. 499.

by selecting pairs in all possible ways from α, β, γ.

The figure of nine points in which two orthic cubics intersect may be regarded as an extension of the orthocentric four-point determined by two equilateral hyperbolas. It is convenient to extend the term *orthocentric* to such a figure. Resuming the results obtained above, we have:

When three of the points of an orthocentric nine-point are a triad of the orthic curves through the nine points, the remaining six points are imaginary, and are the antipoints of the three real points. The centroid of the nine points is the centre of every orthic cubic through them.

It is convenient to speak of a set of orthocentric points determined by a central pencil as a *central* set. Since any three points determine a pencil of orthic cubics of which they are a triad, any three points, with all their antipoints, form a central orthocentric nine-point.

XIV. *The Foci.*

We shall now attack the problem of finding the foci of the orthic cubic. Let us begin with a few words as to the way in which the foci of a curve appear in analysis with conjugate coördinates. The focus of a curve is the intersection of a tangent from one circular point with a tangent from the other circular point. In other words, if the circular rays from a point are tangent to a curve, that point is a focus of the curve. Now the equation of the circular rays from a point a, $\bar{a}$, is

$$(x - a)(\bar{x} - \bar{a}) = 0.$$

Therefore, one of the lines is

$$x - a = 0,$$

and the other is

$$\bar{x} - \bar{a} = 0.$$

Suppose the equation of the curve is

$$F(x\bar{x}) = 0.$$

Now if the circular ray

$$x - a = 0$$

is tangent to the curve, then

$$F(a\bar{x}) = 0,$$

the eliminant of x between these two, will have equal roots. But since the equation of a real curve must be self-conjugate, if this has two coincident roots, then

$$\bar{F}(ax) = 0$$

must also have, and the point a, $\bar{a}$, is a focus. It follows that to find the foci of a curve, we have merely to find those values of x which make two values of $\bar{x}$ coincide. They are the vectors of the foci. Let us apply this method to the orthic cubic. The equation may be taken in the form

$$x^3 - 3x = 2z = a_0 + \lambda a_1$$

where λ is a real parameter and the director line is

$$a_0 + \lambda a_1 = 2z, \quad \bar{a}_0 + \lambda \bar{a}_1 = 2\bar{z}.$$

These relations imply the conjugate expression

$$\bar{x}^3 - 3\bar{x} = 2\bar{z} = \bar{a}_0 + \lambda \bar{a}_1.$$

Two values of $\bar{x}$ become equal when $D_{\bar{x}}\bar{z} = 0$, *i.e.*, when

$$\bar{x}^3 - 1 = 0,$$

or

$$\bar{x} = \pm 1.$$

These values of $\bar{x}$ occur when

$$\bar{a}_0 + \lambda \bar{a}_1 = \pm 2,$$

or

$$\lambda = \frac{-\bar{a}_0 \pm 2}{\bar{a}_1}.$$

Either of these values of λ when substituted in

$$x^3 - 3x = a_0 + \lambda a_1$$

gives three points which are foci of the cubic.

There are, in general, six real foci, which fall into two sets of three. Each set of three corresponds to a single point of the z-plane, and is, therefore, a maximum inscribed triangle of one of the ellipses described above.

XV. *The Foci and the Branch Points.*

If we eliminate the parameter between

$$2z = a_0 + \lambda a_1$$

and

$$2\bar{z} = \bar{a}_0 + \lambda \bar{a}_1,$$

we get the equation of the line ξ,

$$\bar{a}_1 z - a_1 \bar{z} = a_0 \bar{a}_1 - a_1 \bar{a}_0.$$

Now suppose, for a moment, that this line does not contain either of the branch points $z = \pm 1$. Then, if we put $\bar{z} = \pm 1$ in the equation of the line and solve for z, we get a value which is not the conjugate of $\bar{z}$, but is the vector of the reflection of the point $z = \pm 1$ in the line considered. The three points in the x-plane got by putting

$$\lambda = \frac{-\bar{a}_0 \pm 2}{\bar{a}_1}$$

in the equation

$$x^3 - 3x = 2z$$

are the points mapped in the z-plane by the reflection of $z = \pm 1$ in the line ξ. It follows that:

The real foci of the orthic cubic which corresponds to a given line are the six points which correspond to the reflections in that line of the branch points.

If the director line pass through one of the branch points (*i.e.*, if $\frac{-\bar{a}_0 \pm 2}{\bar{a}_1}$ is real), two foci coincide to form the node, and the remaining one of the set is on the curve. One who looks at the matter from the point of view of the Riemann surface might be surprised that a branch point is to be reflected in the line in each sheet of the surface and not in the two sheets alone which it connects. A moment's consideration will show that whether or not

two $\bar{x}$'s coincide depends on λ alone, and that either of three values of x give λ a particular value. It is clear that the reflection must be in every sheet of the surface.

In general, the orthic cubic is of class six. Since it cuts the line at infinity in three points apolar with the circular points, it cannot contain one of the circular points unless it is as a point of inflection. There should, therefore, be six tangents from each of the circular points and, consequently, thirty-six foci. The thirty foci still to be accounted for are the antipoints[1] of the six real foci, paired in all possible ways. When the cubic has a node it is of class four, and has but four real foci. The node takes the place of the two foci which coincide there.

The circular rays

$$x - a_1 = 0$$

and

$$\bar{x} - \bar{a}_2 = 0$$

meet at a_1, $\bar{a}_2$. So the thirty-six foci of an orthic cubic may be represented by the scheme of coördinates:

$$a_i, \bar{a}_j,$$

where i and j run from one to six. It follows that the centroid of the whole thirty-six is the centroid of the six real points; that is, the centre of the cubic.

Consider any selection of three foci. All their antipoints are foci, and the nine points together make up a central orthocentric set.

XVI. *The Foci of the Orthic Cubics which Have a Triad in Common Lie on Two Cassinoids.*

The foci of all the orthic cubics which have a common triad $\alpha\beta\gamma$ lie on two cassinoids which have their foci at α, β, and γ, and are orthogonal to the orthic curves.

We know that these cubics correspond to all the lines through a point, and that their foci correspond to the reflections of the branch points in those lines. Now the reflections of a fixed point in all the lines through a second point lie on a circle which goes through the first point, and which has its centre at the second point. Accord-

[1] Salmon, *Higher Plane Curves*, third edition, p. 122.

ingly, the foci of the cubics will lie on the curves which are the maps in the x-plane of two concentric circles in the z-plane. The centre of these circles maps into the triad common to all the cubics, and the circles themselves map into two cassinoids of the sixth order, about the triad, as M. Lucas[1] has shown. Each of the circles goes through one of the branch points, and, therefore, each of the cassinoids must have a node. If the point which corresponds to the triad $\alpha\beta\gamma$ is equidistant from the branch points, the two circles and also the two cassinoids coincide. In this case the cassinoid has two double points.

The lines which correspond to the cubics are all perpendicular to the circles which correspond to the cassinoids, and so, by the principle of orthogonality, the ovals are orthogonal trajectories of the cubics of the pencil.

XVII. *The Position of the Orthic Cubic in Projective Geometry.*

I shall close this study of the metrical properties of the orthic curve of the third order by showing that from the point of view of projective geometry the orthic cubic is really a general cubic. Any proper plane curve of the third order can be projected into an orthic curve.

We know that the points of contact of three of the six tangents to a cubic curve from any point of its Hessian lie in a line. Now these three points, considered as a binary cubic, have a Hessian pair. If this pair of points be projected to the circular points at infinity, the three tangents become equally inclined asymptotes, and continue to meet in a point. The cubic curve is then orthic and the transformation is accomplished. This projection requires two points to go into given points, and can, therefore, always be made. *In projective geometry the orthic cubic is any proper plane cubic.*

As an illustration of the way in which information about the orthic cubic applies to cubic curves in general, let us see what the characteristic property that the asymptotes are concurrent and equally inclined means. The circular points I and J are a pair of points apolar with the curve. Their join, the line at infinity, meets the curve in three points such that the tangents at these points meet

[1] Felix Lucas, Géométrie des Polynomes, *Journal de l'Ecole Polytechnique*, XXVIII, p. 5.

in a point, C, of the Hessian. Now we know[1] that such a line meets the Hessian in the point which corresponds to C. This leads to the theorems that:

The line joining two points apolar with a cubic curve meets the cubic in three points, the tangents at which meet in a point of the Hessian, and are apolar with the two points apolar with the curve.[2]

The line joining two points apolar with a cubic curve, and a tangent to the cubic at a point of this line, meet the Hessian of the given cubic in corresponding points.

A more novel result is the following. We have seen (XIV, p. 28) that the foci of an orthic cubic fall into two sets of three, in such a way that the two sets are triangles of maximum area inscribed in two confocal ellipses. Now if we consider tangents from I and J instead of foci, we have the following theorem:

If a *and* b *are a pair of points apolar with a cubic curve, then the tangents from either of these points, say* a, *fall into two sets of three in such a way that the line* ab *has the same polar pair of lines as to each set of three.*

Part Two—Orthic Curves of any Order.

I. *Introduction.*

In the preceding pages we have studied the metrical properties of the orthic cubic in some detail. In the following portion of the work I shall indicate an extension of the more important results obtained in the study of the cubic to orthic curves of any order.

The general equation of the n^{th} degree between x and $\bar{x}$ contains $\frac{1}{2}n(n-1)$ product terms. If it is to represent an orthic curve the coefficients of these terms must be made zero. In other words, to make a curve of the n^{th} order orthic is equivalent to making it satisfy $\frac{1}{2}n(n-1)$ linear conditions. After this has been done there remain $2n$ degrees of freedom.

II. *The Orthic Curve is Equilateral.*

The kinematical definition which we obtained for the orthic cubic may be extended to curves of any order, that is:

[1] Salmon, *Higher Plane Curves*, third edition, articles 70 and 175.

[2] "On the Algebraic Potential Curves," Dr. Edward Kasner, *Bulletins of the American Mathematical Society*, June, 1901, p. 393.

The path of a point which moves so that its orientation from n *fixed points is constant is an orthic curve of order* n.

If $a_1, a_2, \ldots a_n$ are the fixed points, the condition on x is expressed by the relations

$$(x - a_1)(x - a_2) \ldots (x - a_n) = \rho\tau_1$$

and

$$(\bar{x} - \bar{a}_1)(\bar{x} - \bar{a}_2) \ldots (x - a_n) = \rho\tau_1^{-1}.$$

These lead to the equation of the curve,

$$x^n - s_1x^{n-1} + s_2x^{n-2} \ldots + \tau_1^2(\bar{s}_n \ldots + \bar{s}_1\bar{x}^{n-1} - \bar{x}^n) = 0,$$

where the s's are the elementary symmetric combinations of the a's. This is the general equation of an orthic curve. If we take $x = \frac{1}{n}s_1$ for a new origin, and make τ_1^2 real, the equation becomes

$$x^n + a_1x^{n-1} - a_2x^{n-2} \ldots - \overline{a_2}x^{n-2} + \overline{a_1}x^{n-1} + \bar{x}^n = 0.$$

The asymptotes are the n equally inclined lines given by the factors of the highest terms,

$$x^n + \bar{x}^n = 0.$$

These lines all pass through the origin; it follows that the centroid of the n points $a_1, \ldots a_n$, is the centre of the curve. Since every orthic curve can be brought to the above form, we see that every orthic curve is equilateral. The converse proposition, every equilateral is orthic, is not true. The general equation of an equilateral may be put in the form

$$x^n + a\bar{x}^n + \Phi(x\bar{x}) = 0,$$

where $\Phi(x\bar{x})$ is a perfectly general function of degree $n-2$. Φ contains $\frac{1}{2}(n-2)(n-3)$ product terms, which must vanish for the curve to be orthic. To make an equilateral curve orthic is, therefore, equivalent to making it satisfy $\frac{1}{2}(n-2)(n-3)$ linear conditions. For $n=2$ and $n=3$ this number is zero, so the equilateral conic and cubic are orthic. For the quartic, this says that to be orthic is one condition.

III. *N-ads, Foci, Intersections with a Circle.*

The relation

$$(x - a_1)(x - a_2) \ldots (x - a_n) = \rho\tau_1 = z$$

may be regarded as mapping a line through the origin in the z-plane into the orthic curve in the x-plane. The methods of analysis which were used, in the paragraphs referred to, in the study of the orthic cubic may be extended to any n, and lead to the following general theorems:

On an orthic curve of order n *there is a single infinity of sets of* n *points, n-ads of the curve, from which all points of the curve have the same orientation. All the n-ads have the same orientation from any point of the curve* (Part One, VIII).

Any n points may be taken as an *n-ad* of an orthic curve. If we take n points of the unit circle as an *n-ad*, and find the remaining intersections of the circle and the curve, we see that they are the vertices of a regular polygon (Part One, XII).

Every circle through an n-ad of an orthic curve of order n *meets the curve again in the* n *vertices of a regular polygon.*

The centre of an orthic curve is the centroid of every n-ad of the curve.

For when the equation is taken in the form

$$x^n + nx^{n-2} + \ldots + a_{n-2}x = z$$

the origin is the centre of the curve, and is also the centroid of the n points which correspond to a point z. This equation will have two coincident roots whenever

$$D_x z \equiv nx^{n-1} + n(n-2)\, a_1 x^{n-3} \ldots = 0.$$

In general, this will give $n-1$ branch points in the z-plane. Each branch point, when reflected in the director line, gives rise to n real foci. If the line ξ revolve about a point, each reflection generates a circle (Part One, XIV). All $n-1$ of these circles are concentric; and they map into $n-1$ cassinoids, on which lie the foci of the curves which have the *n-ad* which corresponds to the centre of the system of circles. These cassinoids are orthogonal trajectories of the central pencil of orthic curves. Since each of the circles must contain a branch point, each cassinoid must have at least one node.

IV. *The Orthic Curve Referred to its Intersections with a Circle.*

We know that we may put $2n$ linear conditions on an orthic curve. If we make it go through $2n$ points on the unit circle, its

equation, expressed in terms of the elementary symmetrical functions of the points where it meets the circle, becomes

$$x^n - s_1 x^{n-1} + s_2 x^{n-2} - + - s_{2n-1}\bar{x}^{n-1} + s_{2n}\bar{x}^n = 0.$$

The centre, found by equating to zero the $n-1^{st}$ derivative as to x, is

$$x = \tfrac{1}{n} s_1.$$

This is the midpoint of the stroke from the centre of the circle to the centroid of the $2n$ points. The equation of an asymptote now takes the form

$$x - \tfrac{1}{n} s_1 = \sqrt[n]{-s_{2n}}\,(\bar{x} - s_{2n-1}\, s_{2n}^{-1}).$$

V. *Construction of an Orthic Curve.*

The method which I have proposed (Part One, V) for the construction of an orthic cubic might be extended to the construction of any orthic curve. For this purpose the instrument must have n hands, moved by n weights. The centre of gravity of any number of weights could be held by joining them together in sets of three or less, and then joining again the centres of gravity of these sets. This operation could be repeated until the required number of weights is reached.

VI. *Geometrical Characteristics.*

The geometrical characteristics of an orthic curve of order n *are that it is equilateral, and that it intersects its asymptotes in points of a second orthic curve of order* n — 2.

For consider the orthic curve referred to its centre,

$$x^n + a_1 x^{n-2} - a_2 x^{n-3} \ldots - \bar{a}_2 \bar{x}^{n-3} + \bar{a}_1 \bar{x}^{n-3} + \bar{x}^n = 0.$$

The asymptotes, which are given by

$$x^n + \bar{x}^n = 0,$$

are concurrent and equally inclined, so the curve is equilateral. The points common to the curve and its asymptotes lie on the curve

$$a_1 x^{n-3} - a_2 x^{n-3} + \ldots - \bar{a}_2 \bar{x}^{n-3} + \bar{a}_1 \bar{x}^{n-3} = 0.$$

But this curve is of order $n-2$, and is orthic.

To require a curve to be equilateral is to impose $2n-3$ conditions, and to require the curve of order $n-2$, along which it cuts its asymptotes, to be orthic is to impose $\frac{1}{2}(n-2)(n-3)$ further conditions, in all $\frac{1}{2}n(n-1)$. But $\frac{1}{2}n(n-1)$ is the number of conditions required to make a curve of order n orthic.

Part Three—Pencils Determined by Two Orthic Curves and Orthocentric Sets of Points.

I. *Introduction.*

We shall now take up the study of the pencils of curves determined by two orthic curves. The main purpose of this investigation shall be to learn what we can about the figure of n^2 points in

Figure 2. The hypocycloid of class five and order six, which is enveloped by the asymptotes of curves in a pencil of orthic cubics.

which two orthic curves intersect. Such a figure of n^2 points we shall call an *Orthocentric Set*, or an *Orthocentric n^2-point*.

There is a well-known proposition that all the equilateral hyperbolas (orthic conics) which can be circumscribed to a given triangle pass through the orthocentre of the triangle. The four points, the vertices and the orthocentre of a triangle, or, what is the same thing, the intersections of two orthic curves of the second order, have the property that the line joining any two of them is perpendicular to the line joining the other two. The term orthocentric is applied

to a set of four points related in this way. We wish to find out what metrical property distinguishes the n^2-point, in which two orthic curves of order n intersect.

II. *The Central Pencil and Its Orthocentric Set.*

The first generalization which we shall make is to show that any pair of points, α, β, together with their antipoints, α, $\bar{\beta}$ and β, $\bar{\alpha}$, form an orthocentric four-point. α and β determine a central pencil of orthic conics,

$$(x-\alpha)(x-\beta)=\tau(\bar{x}-\bar{\alpha})(\bar{x}-\beta),$$

and the antipoints are evidently on all the curves of the pencil.

If we consider τ as a parameter in the general equation of an orthic curve,

$$(x-a_1)(x-a_2)\ldots(x-a_n)=\tau(\bar{x}-\bar{a}_1)(\bar{x}-\bar{a}_2)\ldots(\bar{x}-\bar{a}_n),$$

we obtain the equation of all the curves of which $a_1 \ldots a_n$ is an *n-ad.* The points of the orthocentric n^2-point determined by this are the n real points α, and all their antipoints. But as the pencil is determined by the n real points it follows that:

Any n *points, with all their antipoints, form a central orthocentric n^2-point.*

The centroid of the n^2-point determined in this way is the centroid of the n real points. The real and imaginary foci of any curve are such a set of orthocentric points.

III. *The Pencil of Orthic Cubics through Five Points of a Circle. The Locus of Centres.*

We have seen that six points of a circle determine an orthic cubic curve. If the six points are t_1, t_2, t_3, t_4, t_5, t_6, then, as we have seen, the equation of the orthic cubic through them is

$$x^3-s_1x^2+s_2x-s_3+s_4\bar{x}-s_5\bar{x}^2+s_6\bar{x}^3=0.$$

If we replace t_6 by a variable parameter t, and put σ's for the elementary symmetrical combinations of $t_1 \ldots t_5$, we have

$$s_1=\sigma_1+t, \quad s_4=\sigma_4+t\sigma_3,$$
$$s_2=\sigma_2+t\sigma_1, \quad s_5=\sigma_5+t\sigma_4,$$
$$s_3=\sigma_3+t\sigma_2, \quad s_6=\sigma_5 t.$$

If we make this substitution we get

$$x^3 - (\sigma_1 + t)x^2 + (\sigma_2 + t\sigma_1)x - (\sigma_3 + t\sigma_2)$$
$$+ (\sigma_4 + t\sigma_3)\bar{x} - (\sigma_5 + t\sigma_4)\bar{x}^2 + \sigma_5 t\bar{x}^3 = 0.$$

This is the equation of a pencil of orthic cubics through five points of a circle.

The centre of the curve through the six points is $x = \frac{1}{3}s_1$. If the sixth point move around the unit circle, this becomes

$$x = \tfrac{1}{3}(\sigma_1 + t).$$

This is the map equation of a circle. We have thus the theorem:

The locus of centres of the orthic cubics through five points of a circle is a circle. Its radius is one-third that of the given circle, and its centre is the point $\frac{1}{3}\sigma_1$.

M. Serret[1] gives an elegant synthetic proof of the theorem that the locus of centres of the curves of a pencil of equilaterals is a circle. I obtained the same result for orthic curves independently, and, as the analysis is so direct, it seems advisable to let it stand.

IV. *The Hypocycloid Enveloped by the Asymptotes.*

I shall now prove, for the pencil of orthic cubics through five points of a circle, a theorem which M. Serret[1] states without proof. The theorem referred to, when stated for orthic cubics of the pencil under discussion, becomes:

The curve enveloped by the asymptotes of all the orthic cubics through five points of a circle is an hypocycloid of order six and class five.

It is circumscribed to the centre circle of the pencil, and its cusps lie on a concentric circle five times as large.

We found that the equation of an asymptote, in terms of the six points where the curve cuts the unit circle, is

$$(x - \tfrac{1}{3}s_1) + \sqrt[3]{\sigma_6}(\bar{x} - \tfrac{1}{3}s_5\sigma_6^{-1}) = 0.$$

If we replace t_6 by the parameter t, this becomes

$$x - \tfrac{1}{3}(\sigma_1 + t) + \sqrt[3]{\sigma_6 t}\,(\bar{x} - \tfrac{1}{3}\{\sigma_4\sigma_6^{-1} + t^{-1}\}) = 0.$$

[1] Sur les faisceaux reguliers et les équilatères d'ordre n. Paul Serret, *Comptes Rendus*, 1895, t. 121, pp. 373–5.

we seek the curve enveloped by this line, as t runs around the unit circle.

For the sake of simplicity, let us refer this equation to a new system of coördinates, so chosen that the centre circle of the pencil becomes the new unit circle. The equation becomes

$$x - t + \sqrt[3]{\sigma_0 t}\,(\bar{x} - t^{-1}) = 0.$$

If now we take an axis of reals which makes $\sigma_0 = 1$, and also put τ^3 for t, we have

$$x\tau^{-1} - \tau^3 + \bar{x} - \tau^{-3} = 0.$$

The map equation of the curve enveloped by this line is obtained by equating to zero the result of differentiating with respect to τ. It is

$$x = 3\tau^{-2} - 2\tau^3.$$

This is a curve of double circular motion. The curve is of order six, for it meets any line,

$$x = \frac{a}{1-\tau},$$

where

$$\frac{a}{1-\tau} = \frac{3}{\tau^2} - 2\tau^3,$$

or

$$2\tau^6 - 2\tau^5 - a\tau^2 + 3\tau - 3 = 0.$$

This gives six τ's, and, therefore, the curve is of the sixth order In order to determine the class of the curve, we must examine the equation of a tangent,

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0.$$

This is of the fifth degree in the parameter, and there are, therefore, five tangents from any point x.

The stationary points, or cusps, are the points where the velocity of x is zero. For such a point we must have $D_\tau x = 0$, and at the same time $|\dot{\tau}| = 1$. Both these conditions are satisfied by

$$\tau = \sqrt[5]{-1}.$$

The curve has, therefore, five real cusps; one when τ is each of the fifth roots of minus one.

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If we put $\kappa^5 = -1$ we get a cusp.

$$x = 3\kappa^{-2} - 2\kappa^3,$$

$$\kappa^2 x = 5.$$

Since multiplication by κ^2 is equivalent to a rotation $\frac{4}{5}\pi$, we see that the locus of cusps is a circle, about the centre of the pencil, and five times as large as the centre circle. A rotation $\frac{2}{5}\pi$ sends each cusp into another, and so the cusps are equally spaced along the cusp circle. The intersections of the hypocycloid with the centre circle,

$$x\bar{x} = 1,$$

are obtained by solving $\bar{x} = x^{-1}$ for τ.
We have

$$x = 3\tau^{-2} - 2\tau^3,$$

and

$$\bar{x} = 3\tau^2 - 2\tau^{-3}.$$

The parameters of the points sought are the roots of

$$12\tau^5 - 6\tau^{10} - 6 = 0,$$

or of

$$(\tau^5 - 1)^2 = 0.$$

There are five pairs of coincident intersections. But since x cannot be less than 1, it follows that the curve is tangent to the circle in five places.

We have obtained this hypocycloid as the locus of one asymptote. But all three asymptotes envelop the same curve, for if we put ω for $\sqrt[3]{\sigma_3}$ we get

$$x = 3\omega\tau^{-2} - 2\tau^3.$$

This has a cusp at $\kappa^{12}x = 5$; it is, obviously, the same curve.

V. *Perpendicular Tangents of the Hypocycloid.*

The equation of a tangent to the curve is

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0.$$

That of a perpendicular tangent,

$$-x\tau^{-1} - \tau^3 + \bar{x} + \tau^{-3} = 0.$$

These two lines meet at

$$x = \tau^{-2}.$$

In other words, *Perpendicular tangents to the envelope of the asymptotes meet on the centre circle.*

We have here a verification of the known property of the hypocycloid of this class, that the tangents from a point of the vertex circle are all real and form two regular pencils.[1]

VI. *The Orthocentric Nine-point of the Pencil through Five Points of a Circle, and the Extension to 2n—1 Points.*

Let us now consider the figure of nine orthocentric points, five of which are on a circle. The equation of the pencil of orthic cubics through five points of a circle is

$$x^3 - (\sigma_1 + t)x^2 + (\sigma_2 + t\sigma_1)x - (\sigma_3 + t\sigma_2)$$
$$+ (\sigma_4 + t\sigma_3)\bar{x} - (\sigma_5 + t\sigma_4)\bar{x}^2 + \sigma_5 t\bar{x}^3 = 0.$$

We know five of the points of the orthocentric nine-point determined by this pencil, and we seek the remaining four. Rewrite the above equation as

$$(x - t)\,(x^2 - \sigma_1 x + \sigma_2)$$
$$+ (t\bar{x} - 1)\,(\sigma_3 - \sigma_4\bar{x} + \sigma_5\bar{x}^2) = 0.$$

Now if both

$$x^2 - \sigma_1 x + \sigma_2$$

and

$$\bar{x}^2\sigma_5 - \sigma_4\bar{x} + \sigma_3$$

can become zero for conjugate values of x and $\bar{x}$, then those values are the coördinates of a real point which is on every curve of the pencil, and is one of the nine points. If we put $\sigma_5 = 1$, as we may, these two relations become

$$x^2 - \sigma_1 x + \sigma_2 = 0,$$

[1] F. Morley, "On the Epicycloid," *American Journal of Mathematics*, Vol. XIII, No. 2.

and

$$\bar{x}^2 - \bar{\sigma}_1 x_1 + \bar{\sigma}_2 = 0.$$

These are conjugate equations and so can be satisfied by the coordinates of real points. Solving them, we get a pair of real points

$$x_1 = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}, \quad \bar{x}_1 = \frac{\bar{\sigma}_1 + \sqrt{\bar{\sigma}_1^2 - 4\bar{\sigma}_2}}{2};$$

and

$$\frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2}, \quad \bar{x}_2 = \frac{\bar{\sigma}_1 - \sqrt{\bar{\sigma}_1^2 - 4\bar{\sigma}_2}}{2}.$$

But further, we notice that the antipoints, x_1, $\bar{x}_2$, and x_2, $\bar{x}_1$, of these make the equation of the pencil vanish for all values of the parameter. They are the remaining points of the orthocentric nine. This leads to the theorem that:

If five points of an orthocentric nine-point are on a circle, of the remaining four points two are real, two are imaginary; and these four form an orthocentric four-point.

The centroid of the nine points is

$$x = \tfrac{1}{9}(t_1 + \ldots t_5 + 2\sigma_1) = \tfrac{1}{3}\sigma_1.$$

This is the centre of the centre circle of the pencil.

We can extend these results to the case of n^2-points, $2n - 1$ of which lie on a circle.

The pencil of orthic curves of order n which go through $2n - 1$ points of the unit circle is given by

$$x^n - (\sigma_1 + t)\, x^{n-1} + (\sigma_2 + t\sigma_1)\, x^{n-2} - + \ldots$$

$$(\sigma_{2n-2} + t\sigma_{2n-3})\, \bar{x}^{n-2} - (\sigma_{2n-1} + t\sigma_{2n-2})\, \bar{x}^{n-1} + \sigma_{2n} t \bar{x}^n = 0.$$

If we let $\sigma_{2n-1} = 1$, this becomes

$$(x - t)\,(x^{n-1} - \sigma_1 x^{n-2} + \sigma_2 x^{n-3} - + - \ldots \sigma_{n-1})$$

$$+ (\bar{x}t - 1)\,(\bar{x}^{n-1} - \bar{\sigma}_1 \bar{x}^{n-2} + - \ldots \bar{\sigma}_{n-1}) = 0.$$

Now since the coefficients of $(x - t)$ and $(\bar{x}t - 1)$ are conjugate forms, there are $n - 1$ real points, in addition to the points t_1, t_2 $\ldots$ t_{2n-1}, which are on all the curves of the pencil. Further,

all the antipoints obtained by pairing these in all possible ways satisfy the equation for all values of t. Now we know that the $(n-1)^2$ points thus found form an orthocentric set. We are now in a position to state the following general theorem:

If $2n-1$ points of an orthocentric set of n^2-points lie on a circle, then the remaining $(n-1)^2$ points of the figure form a central orthocentric set of which $n-1$ points are real.

The vectors of the $n-1$ real points are the roots of

$$x^{n-1} - \sigma_1 x^{n-2} + \sigma_2 x^{n-3} - + \ldots \sigma_{n-1} = 0.$$

VII. *The Pencil Determined by Any Two Orthic Curves.*

We are now ready to consider the most general pencil of orthic curves. Form the equation

$$x^n - (a_1 + ta'_1)\, x^{n-1} + (a_2 + ta'_2)\, x^{n-2} - + \ldots$$
$$- (a_{2n-1} + ta'_{2n-1})\, \bar{x}^{n-1} + t\bar{x}^n = 0,$$

where t is a parameter which has the absolute value unity. Now for every value of t this represents a real orthic curve of the n^{th} order, provided

$$\overline{a_v + ta'_v} = (a_{2n-v} + ta'_{2n-v})t^{-1},$$

or

$$|\, \bar{a}_v - a'_{2n-v}\,| = |\, \bar{a}'_v - a_{2n-v}\,|.$$

For if this holds, the equation can be put in the known form

$$(x-a_1)(x-a_2)\ldots = \tau_1 (\bar{x}-\bar{a}_1)(\bar{x}-\bar{a}_2)\ldots(\bar{x}-\bar{a}_n).$$

Now let

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - + \ldots - a_{2n-1}\bar{x}^{n-1} + a_{2n}\bar{x}^n = 0,$$

and

$$x^n - a'_1 x^{n-1} + a'_2 x^{n-2} - + \ldots - a'_{2n-1}\bar{x}^{n-1} + a'_{2n}\bar{x}^n = 0,$$

be the equations of any two real orthic curves. Then

$$a_{2n} = t_1,\ a'_{2n} = t_2,$$

and

$$\bar{a}_v = a_{2n-v} t_1^{-1},\ \bar{a}'_v = a'_{2n-v} t_2^{-1}.$$

We can choose the a's in such a way that the pencil will include the given curves, (1) and (2), for the $4n - 2$ equations

$$a_v = a_v + t_1 a'_v,$$

$$a'_v = a_v + t_2 a'_v, \qquad v = 1 \ldots 2n-1$$

just suffice. We must show now that when the coefficients are determined as above, all the curves of the pencil are real.

Now we have

$$a_v = a_v + t_1 a'_v,$$

and

$$a_{2n-v} = a_{2n-v} + t_1 a'_{2n-v}.$$

From these, we get

$$\overline{a_v + t_1 a'_v} = \bar{a}_v = a_{2n-v} t_1^{-1} = a_{2n-v} t_1^{-1} + a'_{2n-v},$$

and therefore

$$\bar{a}_v - a'_{2n-v} = (a_{2n-v} - \bar{a}'_v) t_1^{-1}.$$

But this is the condition that every curve of the pencil be real. It is clear that no curve not orthic can be included in the pencil. So we see that:

Any two real orthic curves of order n *determine a pencil of real curves of the same order, all of which are orthic.*

VIII. *The Serret Circle, or Locus of Centres.*

M. Serret's theorem (Part Three, IV) on the locus of centres is easily verified. The centre of any curve of the pencil is

$$x = \tfrac{1}{n}(a_1 + t a'_1).$$

Now if t is regarded as a parameter, this is the map equation of a circle with its centre at

$$x = \tfrac{1}{n} a_1.$$

The locus of centres of the most general pencil of orthic curves is a circle.

In the special case where n of the intersections of the pencil are at infinity, the locus of centres degenerates into a right line. A pencil of this type may be written

$$x^n - \frac{a_1 - \lambda a'_1}{1 - \lambda} x^{n-1} + \ldots \frac{a_{2n-1} - \lambda a'_{2n-1}}{1 - \lambda} \bar{x}^{n-1} + \bar{x}^n = 0,$$

where λ is a real parameter. The locus of centres is

$$x = \tfrac{1}{n}\left(\frac{a_1 - \lambda a'_1}{1 - \lambda}\right).$$

The elimination of λ from this and its conjugate gives

$$x\,(a_{2n-1} - a'_{2n-1}) - \bar{x}\,(a_1 - a'_1) + \tfrac{1}{n}\,(a_1 a'_{2n-1} - a'_1 a'_{2n-1}) = 0,$$

the equation of a right line.

IX. *The Hypocycloid Enveloped by the Asymptotes.*

Let us now seek the curve enveloped by the asymptotes of the curves of a general pencil. The equation of an asymptote of the curve given by t_1 is

$$x - \tfrac{1}{n}\,(a_1 + t_1 a'_1) + \sqrt[n]{t_1}\left\{\bar{x} - \tfrac{1}{n}\,(a_{2n-1} t^{-1} + a'_{2n-1})\right\} = 0,$$

or

$$x - \tfrac{1}{n}\,(a_1 + t_1 a'_1) + \sqrt[n]{t_1}\left\{\bar{x} - \overline{(a_1 + t_1 a'_1)}\right\} = 0.$$

For convenience, transform to the centre of the pencil, $\frac{1}{n}a$, as a new origin. The equation becomes

$$x - \tfrac{1}{n}\,a'_1 t_1 + \sqrt[n]{t_1}\,(x - \tfrac{1}{n}\overline{a'_1 t_1}) = 0.$$

Putting $\tau^n = t$, we get

$$x - \tfrac{1}{n} a'_1 \tau^n + \tau \bar{x} - \tfrac{1}{n}\bar{a}'_1 \tau^{1-n} = 0,$$

and finally,

$$x\tau^{-1} - \tfrac{1}{n} a'_1 \tau^{n-1} + \bar{x} - \tfrac{1}{n}\bar{a}'_1 \tau^{-n} = 0.$$

Now the map equation of the curve enveloped by this line as τ varies is

$$nx = n\bar{a}'_1 \tau^{1-n} + (1 - n)\, a'_1 \tau^n.$$

Now this equation represents a curve of double circular motion. We know that

$$\bar{a}'_1 = a'_1 t_2^{-1},$$

and using it we get

$$nx = n a'_1 t_2^{-1} \tau^{1-n} + (1 - n)\, a'_1 \tau^n.$$

Now if we make t_2 real, and then regard the centre circle as the unit

circle, *i.e.*, adopt $\left|\frac{a'_1}{n}\right|$ as the unit length, the equation takes the form

$$x = n\tau^{1-n} + (1 - n)\tau^n.$$

This is the equation of an hypocycloid of the kind found as the locus of asymptotes of a special pencil of orthic cubics. Its vertex circle is the centre circle of the pencil. It has cusps when

$$D_\tau x = 0,$$

and

$$|\tau| = 1,$$

simultaneously, or when

$$\tau^{2n-1} + 1 = 0.$$

The parameters of the cusps are the $2n - 1^{st}$ roots of -1.
If we let $\kappa^{2n-1} = -1$, a cusp is

$$x = n\kappa^{1-n} + (1 - n)\kappa^n$$

or

$$x\kappa^{n-1} = n - (1 - n)$$

The absolute value of a cusp is, therefore, $2n - 1$.

Since the equation of a tangent,

$$x - \tfrac{1}{n}a'_1\tau^n + \tau\bar{x} - \tfrac{1}{n}\tau^{1-n}\bar{a}'_1 = 0$$

is of the $2n - 1^{st}$ degree in the parameter τ, the hypocycloid is of class $2n - 1$. If we eliminate x between the equation of the curve and the equation of any line,

$$x = \frac{a}{1 - \tau}$$

we get an equation of the $2n^{th}$ degree to determine the parameters of the points of intersection. The curve meets any line in $2n$ points, and is therefore of order $2n$. We have now established analytically the theorem stated by M. Serret, as far as orthic curves are concerned. It is:

The curve enveloped by the asymptotes of a pencil of orthic curves of order n *is an hypocycloid of order* 2n, *and of class* 2n — 1. *Its vertex circle is the centre circle of the pencil, and its cusp circle is concentric with that circle, and* 2n — 1 *times as large.*

If we bear in mind that any difference between an orthic curve

and any equilateral does not affect the terms of the n^{th} and $n-1^{st}$ degrees of the equation, we see that the method of proof used above is applicable to equilaterals in general.

X. *A Circle Determined by Any Odd Number of Points.*

It is a well-known proposition that the centres of the equilateral hyperbolas circumscribed to a triangle lie on the circle through the mid-points of the sides of the triangle. This circle is usually called the Feuerbach, or nine-point circle of the triangle. Now we have seen that an orthic curve of order n may be made to satisfy $2n$ linear conditions; it follows that any odd number, $2n-1$, of points determine a pencil of orthic curves of the n^{th} order. Connected with this pencil is the centre circle, or, as I propose to call it, the Serret circle, which is, in a sense, the generalized nine-point circle.

Every figure of an odd number of points has connected with it a unique circle, the Serret circle, which in the case of three points is identical with the nine-point circle of Feuerbach.

Further, every odd number of points, $2n-1$, determine the pencil of orthic curves through them, and therefore the remaining $(n-1)^2$ points of the orthocentric n^2-point. In the case of three given points, this set of $(n-1)^2$ points is a single point, the orthocentre of the given points. So we are led to the theorem:

To every figure of $2n-1$ *points belongs a figure of* $(n-1)^2$ *points.*

In one sense the Serret circle belongs to n^2 points, but of these only $2n-1$ may be taken at random.

XI. *A Point Determined by An Even Number of Points.*

Now consider an even number, $2n$, of points which do not belong to an orthocentric n^2-point. There is a pencil of orthic curves through every $2n-1$ points which can be selected from them, or $2n$ pencils in all. Now these pencils give rise to $2n-1$ Serret circles, but there is one orthic curve through all $2n$ points and its centre is on each of the circles. We have, therefore, the result:

The $2n$ *Serret circles, given by all the sets of* $2n-1$ *among* $2n$ *points, meet in a point.*

XII. *The Relation of the Orthocentric* n^2*-point to the Circle of Centres.*

In section VIII we obtained the pencil of orthic curves determined by the two given curves,

$$(1)\quad x^n - a_1x^{n-1} + a_2x^{n-2} - + \ldots - a_{2n-1}\bar{x}^{n-1} + a_{2n}\bar{x}^n = 0,$$

and

$$(2)\quad x^n - a'_1x^{n-1} + a'_2x^{n-2} - + \ldots - a'_{2n-1}\bar{x}^{n-1} + a'_{2n}\bar{x}^n = 0.$$

We now wish to show that the centroid of the orthocentric n^2-point in which these two curves intersect is the centre of the centre circle of the pencil. If we rewrite (1) and (2) in terms of $\bar{x}$ we get

$$(1)\qquad (\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_2) \ldots (\bar{x} - \bar{x}_n) = 0,$$

and

$$(2)\qquad (\bar{x} - \bar{x}'_1)(\bar{x} - \bar{x}'_2) \ldots (\bar{x} - \bar{x}'_n) = 0.$$

If the s's refer to the elementary symmetrical functions of the roots, we have

$$s_1 = a_{2n-1},\ s'_1 = a'_{2n-1},\ i = 1, 2, \ldots n - 1.$$

$$s_n = -(x^n - a_1x^{n-1} + a_2x^{n-2} \ldots a_n)\, a_{2n}^{-1},$$

$$s'_n = -(x^n - a'_1x^{n-1} + a'_2x^{n-2} \ldots a'_n)\, a'^{-1}_{2n}.$$

Now the eliminant of x between these two equations is

$$\begin{array}{c}
(\bar{x}_1 - \bar{x}'_1)(\bar{x}_1 - \bar{x}'_2) \ldots (\bar{x}_1 - \bar{x}'_n). \\
(\bar{x}_2 - \bar{x}'_1)(\bar{x}_2 - \bar{x}'_2) \ldots (\bar{x}_2 - \bar{x}'_n). \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
(\bar{x}_n - \bar{x}'_1)(\bar{x}_n - \bar{x}'_2) \ldots (\bar{x}_n - \bar{x}'_n) = 0.
\end{array}$$

This is a function of degree n^2 in x, and as x occurs in s_n and s'_n alone, we need consider those terms alone in which the products s_n and s'_n appear. These are:

$$s_n^n - ns_n^{n-1}s'_n + \frac{n.n-1}{2}\, s_n^{n-2}s'^2_n \ldots s'^n_n,$$

or

$$(s_n - s'_n)^n;$$

or, in terms of x,

$$\left\{x^n\left(\frac{a_{2n} - a'_{2n}}{a_{2n}a'_{2n}}\right) - x^{n-1}\left(\frac{a_{2n}a'_1 - a'_{2n}a_1}{a_{2n}a'_{2n}}\right) + \ldots \left(\frac{a_{2n}a'_n - a'_{2n}a_n}{a_{2n}a'_{2n}}\right)\right\}^n.$$

When this is expanded and arranged in powers of x the first and second terms are

$$x^{n^2}\left(\frac{a_{2n}-a'_{2n}}{a_{2n}a'_{2n}}\right)^n - n\left(\frac{a_{2n}-a'_{2n}}{a_{2n}a'_{2n}}\right)^{n-1}\left(\frac{a_{2n}a'_1-a'_{2n}a_1}{a_{2n}a'_{2n}}\right)x^{n^2-1}.$$

Now the sum of the roots is

$$\sigma_1 = n\,\frac{a_{2n}a'_1-a'_{2n}a_1}{a_{2n}-a'_{2n}},$$

and their centroid is

$$\frac{1}{n^2}\,\sigma_1 = \frac{1}{n}\,\frac{a_{2n}a'_1-a'_{2n}a_1}{a_{2n}-a'_{2n}} = x'.$$

Now $a_{2n} = t_1$, and $a'_{2n} = t_2$, and we have also the relations

$$a_\nu = a_\nu + t_1 a'_\nu,$$

and

$$a'_\nu = a_\nu + t_2 a'_\nu,$$

from which we obtain

$$x' = \tfrac{1}{n}\,\frac{t_1a_1 + t_1t_2a'_1 - t_2a_1 - t_1t_2a'_1}{t_1-t_2}$$

$$= \tfrac{1}{n}a_1.$$

But this is precisely the centre of the centre circle

$$x = \tfrac{1}{n}\,(a_1 + ta'_1).$$

We are thus enabled to conclude with the general theorem:

The centroid of an orthocentric set of points is the centre of the centre circle of the pencil of orthic curves through those points.

Johns Hopkins University, May 20, 1904.

PALLADIUM (Pd).

BY JOSEPH WHARTON, Sc.D., LL.D.

(*Read April 7, 1904.*)

Although palladium belongs to the platinum group of metals, it is in some respects nearly related also to silver, its atomic weight and specific gravity being respectively about 107 and 11.4, while the corresponding figures for silver are 108 and 10.5. In its high melting point, however, of 1500° C., it approaches more nearly to platinum which melts at 1750° C., and in color its grayish-white resembles the color of platinum more nearly than that of silver.

Palladium has long been known to occur native in company with platinum, and also alloyed with gold in the Brazilian mineral porpezite which contains about 5 to 10 per cent. of it. That it occurs in notable quantity in the nickeliferous pyrrhotite of Canada is an important recent observation.

Both platinum and palladium probably exist to a greater or less extent in all the many deposits of nickeliferous pyrrhotite throughout the world; certainly in those of Norway and Sweden, and particularly in every one of the numerous deposits of that mineral which are found in the Laurentian and Huronian rocks surrounding the little town Sudbury, in the province of Ontario, Canada. The quantity, however, is extremely small, varying from a mere trace to one or more ounces per ton; the average for each metal being about one-hundredth of an ounce per ton of ore, platinum and palladium usually being present in approximately equal parts.

Yet, though known to exist in many parts of the world, palladium has not been diligently sought for, because there was until recently no considerable demand for it; the reworking of platiniferous residues from the mints of several countries having supplied most of that which appeared in commerce. The prevailing scarcity of platinum is now directing attention to palladium as a practicable substitute for some purposes.

The nickeliferous pyrrhotite deposits of the Sudbury region have recently become the most important source of nickel in the world and appear certain to continue so for many years, having quite surpassed in yield the great nickel-silicate ores of New Caledonia which come next in rank. In these Canadian ores, silver, gold, platinum, iridium and rhodium occur as well as palladium; all in very minute quantities—palladium as above mentioned to the extent of about .01 oz. per ton.

The form in which palladium there occurs has not been detected, for owing to its minute quantity and the consequent difficulty of isolating it, none has yet been directly observed in any ore of that region; since, however, platinum occurs there as arsenide in the interesting mineral sperrylite ($PtAs_2$), palladium may exist in similar combination, though none has been observed in any specimen of sperrylite that has been examined. Prof. Horace L. Wells indeed notes a trace of palladium in sperrylite, but this has not I think been confirmed in any of the careful analyses of other good chemists.

In the one mine where platinum-arsenide has been found, Prof. Wells says it occurs in small pockets of decomposed rock at the contact of ore and rock, these pockets being filled with loose gravelly material. It was in the metallic sparkles of that sandy stuff where the sun's rays struck it that Mr. Sperry first noticed what proved to be platinum-arsenide—a substance till then unknown.

Ten years ago, when I visited the mine in question, the Vermillion Mine, I observed, upon the surface of the ground where the ore had been dumped, a moderate quantity of sand which appeared to have resulted from the disintegration and metasomatism of ore by atmospheric penetration, and this seems to afford a plausible explanation how palladium-arsenide might have been present in the ore with platinum-arsenide and yet no palladium be now detectible in the Sperrylite; for the greater oxidability of palladium may have led to the conversion of its arsenide into arseniate, afterward leached away by the percolating water.

Though the Vermillion Mine is not at present in operation, it affords, as above stated, the only indication we have as to the probable condition of both platinum and palladium in the ores of other mines in that region from which those metals are extracted, though neither metal has been directly observed in any of those other mines.

The ore from those other mines is not simply nickeliferous pyrrhotite, but is also to a considerable extent chalcopyrite, yielding therefore much copper, as well as nickel and minute proportions of the above-named precious metals. The ores of the various mines in the Sudbury region may be reckoned as containing 1½ to 8 per cent. of nickel (some small quantities even 30 to 40 per cent.) and 1 to 4 per cent. of copper.

Those ores are roasted in open heaps and then smelted into matte containing by average about 30 per cent. of nickel and copper,

and containing also practically all of the precious metals which the ores carried.

After further roasting and smelting, the concentrated matte is treated for separation of copper from nickel, which is effected by repeated melting with nitre cake and coke in cupola furnaces. The coke converts the nitre cake into sodium sulphide; when the charge is run out of the furnace and cooled it separates easily into two parts, the bottoms containing practically all the nickel, the tops consisting of sodium sulphide and copper sulphide; the gold and silver going with the tops, the platinum-group metals going with the bottoms.

In the refining processes that follow, palladium is obtained as a slime, carrying about a thousand times as much palladium proportionally as did the original ore, carrying also the other platinum-group metals, and the gold and silver.

This palladium-bearing slime is melted and refined in a small reverberatory furnace, from which it is ladled out into cold water, forming shot which are charged into small leaden towers, into the top of which hot dilute sulphuric acid is run. Palladium and the other precious metals being electro-negative to the base metals, a galvanic action now takes place in which nickel, copper and iron dissolve rapidly, leaving palladium in a black mud containing two per cent. or more of that metal. If this residue still contains much copper, that is mostly eliminated by further treatment with hot sulphuric acid until the stuff contains about 25 per cent. of palladium, when it is treated with aqua-regia, thus dissolving all the platinum, palladium and gold.

From this solution platinum is precipitated by ammonium chloride, the palladium in the filtrate is electrolytically precipitated with a platinum anode, appearing as a dull gray metal which is hard and brittle, peeling off easily from the cathode. It is then dried and ignited in a reducing atmosphere, when it takes great brilliancy and becomes very soft and pliable, capable of being worked into any ordinary form. I have, for instance, a remarkably nice teaspoon made of it.

If air is not completely excluded during ignition the palladium will oxidize on the surface taking most beautiful colorations of pink and green. When prepared as above stated palladium is almost absolutely pure, but for occasional traces of copper and iron.

I purposely refrain from giving all details of the various stages of

the processes by which pure palladium is finally attained, but enough is stated to show that a complete working system is established, requiring, of course, delicacy of perception and dexterous manipulation, yet yielding at last a beautiful substance capable of sundry uses, which are undeveloped only because no regular supply could hitherto be counted on. The steady production of palladium by the Orford Copper Co. is now more than 3000 ounces annually, from approximately 300,000 tons of Canadian ores treated. It is obvious that only as a by-product in the working of very great quantities of ore can palladium be produced as here stated.

Besides having so very high a melting point, and being at the same time both hard, ductile and malleable, palladium is so absolutely non-corrodible that a sheet of it may hang for a long time in a laboratory exposed to chlorine and hydrogen-sulphide gases without losing its polish or tarnishing.

The wonderful occlusion or absorption of hydrogen by palladium deserves special attention and invites further study.

The volume of hydrogen thus absorbed varies greatly under different circumstances, and has been variously stated by different observers. According to Graham (*Phil. Mag.*, [4] 32, 401, 503):

Fused palladium at 200°				absorbs	68 volumes.
Finely divided palladium at 200°				"	686 "
Sheet palladium at ordinary temperatures (after ignition)				"	376 "
"	"	" 90° to 97°	"	"	643 "
"	"	" 245°	"	"	526 "

In Poggendorff's *Annalen* for 1869, Graham describes experiments in which 900 volumes were absorbed.

The greatest absorption observed before the experiments of McElfresh (mentioned below) was shown by electrolytically precipitated palladium; the maximum being 982.14 volumes of hydrogen.

Schmidt (*Ann. Physik.*, iv, 13,747), *J. Chem. Soc.*, 85, 86, 312 (1904), finds that the volume of hydrogen absorbed by palladium increases with the fall of temperature to about 140°; below this he finds concordant results. From 140° to 300° the absorption curve approaches a straight line. Absorption, and also diffusion, increases with pressure as well as with temperature.

Hoitsema (*Zeit. physikal. Chemie.*, 1895, 17, 1), *J. L. C. S.*, 78, 11,388 (1895) examines the hypothesis of Troost and Hauteville, that in the absorption of H by Pd a compound is formed repre-

sented by Pd_2H, but he does not agree with those authors. Debray thought the compound Pd_4H_2 was formed. Why should a chemical formula be sought for the compound of palladium and hydrogen since they combine together in practically all proportions, thus indicating it to be a simple alloy?

McElfresh (*Proc. Am. Acad. Arts and Sciences*, vol. xxxix, No. 14, Jan., 1904), examining critically the influence of occluded hydrogen upon the electrical resistance of palladium, finds that resistance to increase constantly, but not at a uniform rate, as the occluded hydrogen increases; the maximum increase of the resistance being 68 per cent. when 1030 volumes were occluded. This absorption, reached by continuous exposure for 30 hours, is the highest yet observed and probably indicates complete saturation.

McElfresh considered Knott's method (*Proc. Roy. Soc. Edin.*, vol. xii, 1882, 1883) of determining the amount of occlusion, by measuring the increase in weight of the palladium treated, to be incapable of accuracy; he was also dissatisfied with the results obtained with imperfect apparatus by supplying to palladium a measured quantity of electrolytically produced hydrogen and deducting therefrom the quantity remaining after various periods of absorption. He therefore avoided the error inherent in such apparatus by using in this latter method ingenious apparatus of his own devising, thus reaching conclusions which appear quite reliable.

It is remarkable that Richter's *Chemistry*, as translated by Edgar F. Smith, states (p. 46) that the conductivity of palladium for both heat and electricity is little affected by its occlusion of hydrogen.

As for the discharge of occluded hydrogen from palladium, Graham states that "the gas exhibits no disposition to leave the metal and escape into a vacuum at the temperature of its absorption." Edgar F. Smith informs me that he finds charged palladium immersed in water at 160° to give off hydrogen with freedom comparable to the escape of carbonic acid from soda water.

Baskerville informs me that, in examining palladium for radioactivity, he found none in either of the two forms I sent him at his request for that purpose—namely, electrolytically deposited scale such as mentioned above in this paper, and similar scale which had been fused into a large button. But when he examined the same specimens after charging them in a finely divided state with hydrogen, he found slight indications of radio-activity in the first, but none in the second. He therefore asks whether, during the electro-deposition, a tension might have accumulated which

appeared afterward as beta-rays. Query: If so, why should the electro-scale require charging with hydrogen to enable it to indicate radio-activity?

Sir William Ramsay, to whom I mentioned this experiment and surmise, suggested the possibility that the original ore might have contained a trace of radium, which persisted with the palladium until the final fusion vaporized it or passed it to the slag.

Among the chemical characteristics of palladium may be mentioned these, not of course as novelties but as practically useful:

1. It is completely precipitated from an acid solution as sulphide by hydrogen-sulphide.

2. It is thrown down as a black precipitate from even a dilute solution by potassium-iodide. This very sensitive reaction is important in practical treatment of material containing palladium.

3. It is precipitated by mercury-cyanide as white slimy palladium-cyanide: a property useful for quantitative determination in laboratory.

Among the uses hitherto of palladium are:

1. For the mechanism of delicate instruments such as chronometers, and for verniers, etc., of astronomical instruments.

2. For surgical instruments.

3. For plating searchlight mirrors. Why not for the mirrors of reflecting telescopes?

4. For alloying with silver to make dental plates, etc., instead of the $\frac{2}{3}$ silver, $\frac{1}{3}$ platinum hitherto used in Europe. Also as palladium amalgam for fillings in cavities of teeth.[1]

Other uses will naturally arise as men's minds are turned toward this metal, which, while in many respects equal to platinum, sells for no more than the price by weight of that metal, and of course

[1] Palladium amalgam has been used to but very small extent for tooth fillings, though well adapted for that use except for its dark color, arising apparently from palladium black being used to form the amalgam, which is made by the dentist at the moment he wishes to use it, by triturating palladium black with mercury. That darkness of color might probably be obviated by using fine palladium filings instead of palladium black.

Dr. Joseph Pettit, a careful observer, told me that he had found this amalgam so made to become too hot to be comfortably held in the hand. Dr. W. Storer How, of the S. S. White Dental Co., informed me that he had noticed some warmth evolved in the making of the amalgam. I observed very little heat, no more in fact than I thought referable to the friction of trituration.

therefore for much less than that by bulk; the specific gravity of platinum being variously stated as 17 to 19, and that of palladium as 11.4 to 11.8.

It would seem that palladium might be useful under some circumstances for resistance wire.

I conclude by remarking that, in the several reports by the Canadian Government upon the metallic and mineral resources of "The Dominion," palladium is never mentioned; not even in the report for 1904.

Stated Meeting, April 15, 1904.

President SMITH in the Chair.

Prof. Edward Potts Cheyney and Dr. Harvey W. Wiley, newly elected members, were presented to the Chair, and took their seats in the Society.

Letters accepting membership were read from:

President Roosevelt, Washington, D. C.
Prof. Maurice Bloomfield, Baltimore.
Prof. Henry Pickering Bowditch, Jamaica Plain, Mass.
Prof. Edward Potts Cheyney, Philadelphia.
Prof. Russell H. Chittenden, New Haven.
Prof. Frank Wigglesworth Clarke, Washington.
Prof. Kuno Francke, Cambridge, Mass.
Prof. Edward Leamington Nichols, Ithaca.
Prof. Samuel W. Stratton, Washington.
Dr. Harvey W. Wiley, Washington.

Prof. C. L. Doolittle discussed a letter from Mr. Germers on the Aurora Borealis, and explained the phenomena referred to therein.

The following papers were read:

"The Trail of the Golden Dragon," by Dr. George B. Gordon.

"Views of Old Philadelphia," by Mr. Julius F. Sachse.

Stated Meeting, May 6, 1904.

President SMITH in the Chair.

Letters accepting membership were read from:
Gen. A. W. Greely, Washington.
Prof. P. A. Lambert, Bethlehem.
Prof. Edgar Odell Lovett, Princeton.
Prof. Friedrich Delitzsch, Berlin.
Sir Richard C. Jebb, Cambridge.
Prof. Ernest Rutherford, Montreal.
Prof. J. H. Van't Hoff, Berlin.
Prof. Wilhelm Waldeyer, Berlin.

The decease was announced of Sir Henry Thompson, Bart., at London, on April 18, 1904, æt. 84.

The following papers were read:

"The Vegetation of the Island of Dominica," by Prof. Francis E. Lloyd, who was introduced by the President.

"Orthic Curves; or Algebraic Curves which Satisfy La-Place's Equation in Two Dimensions," by Mr. Charles Edward Brooks, presented by Prof. George F. Barker (see page 294).

Stated Meeting, May 20, 1904.

President SMITH in the Chair.

A letter accepting membership was read from Dr. J. Chalmers DaCosta, Philadelphia.

The decease was announced of:

M. Otto von Böhtlingk, at Leipzig, on April 1, 1904, æt. 89.

Dr. Roberts Bartholow, at Philadelphia, on May 10, 1904, æt. 72.

The following papers were read:

"Recent Researches on Inscriptions from Nippur," by Dr. Albert T. Clay, presented by the President.

"Observations on the New Metallic Salts," by Dr. Edgar F. Smith (by title).

Stated Meeting, October 7, 1904.

President SMITH in the Chair.

Dr. J. Chalmers DaCosta, a newly-elected member, was presented to the Chair, and took his seat in the Society.

The decease of the following members was announced:

Prof. William Henry Pettee, at Ann Arbor, Mich., on May 26, 1904, æt. 66.

Dr. Hermann Rollett, at Baden bei Wien, on May 30, 1904, æt. 85.

Prof. John B. Hatcher, at Pittsburgh, on July 4, 1904, æt. 45.

Alfred H. Allen, F.C.S., at Sheffield, Eng., on July 14, 1904, æt. 60.

Dr. Isaac Roberts, at Crowborough, Sussex, Eng., on July 17, 1904, æt. 75.

Hon. Robert E. Pattison, at Philadelphia, on August 1, 1904, æt. 54.

Matthew Huizinga Messchert, at Douglassville, Pa., on August 23, 1904, aet. 75.

Rev. Dr. Charles W. Shields, at Newport, R. I., on August 26, 1904, æt. 80.

The following papers were read:

"Is Abraham Historical?" by Dr. Albert T. Clay, presented by the President.

"The Morphological Superiority of the Female Sex," by Thomas H. Montgomery, Jr.

"The Morphology of the Skull of the Pelycosaurian Genus, Dimetrodon," by Prof. E. C. Case, presented by the Secretaries.